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Inventory model of type (s, S) under heavy tailed demand with infinite variance

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Abstract. In this study, a stochastic process $X(t)$, which describes an inventory model of type (s, S) is considered in the presence of heavy tailed demands with infinite variance. The aim of this study is observing the impact of regularly varying demand distributions with infinite variance on the stochastic process $X(t)$. The main motivation of this work is, the publication by Geluk [*Proceedings of the American Mathematical Society* **125** (1997) 3407–3413] where he provided a special asymptotic expansion for renewal function generated by regularly varying random variables. Two term asymptotic expansion for the ergodic distribution function of the process $X(t)$ is obtained based on the main results proposed by Geluk [*Proceedings of the American Mathematical Society* **125** (1997) 3407–3413]. Finally, weak convergence theorem for the ergodic distribution of this process is proved by using Karamata theory.

1 Introduction

Over the last decades investigation of heavy tailed distributions, which tend to produce outlying values, has become an important research area. These distributions are used for modeling many physical and economic systems such as medical sciences, civil engineering applications, meteorology, financial risk management and inventory systems. Moreover, they are important tools for studying the properties of such models. The aim of this study is to investigate the impact of regularly varying distributions, which is one of the broad subclass of heavy tailed distributions, on the stochastic process describing an inventory model of type (s, S). Particularly, we obtained our asymptotic results by assuming that the demand random variables belongs to the class of regularly varying random variables with tail parameter $1 < \alpha < 2$.

The problem addressed in this study is based on investigation of a semi-Markovian inventory model of type (s, S) like many other problems in stock control, queuing, stochastic finance and reliability theory. There are numerous studies

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in current literature which deal with the probability and numerical characteristics of a semi-Markovian inventory model of type (s, S) (for example, see Brown and Solomon (1975), Chen and Zheng (1997), Sahin (1983)). In most of these studies, some real world problems have been solved by using dynamic programming but analytic solutions could not be obtained. In order to estimate the behavior of a system, it is important to investigate stationary characteristics, for example ergodic moments and ergodic distribution function. One of the most popular methods to obtain useful formulas for mentioned characteristics is using asymptotic expansion method. In recent years, many researchers have extensively studied the characteristics of an inventory model type (s, S) by using asymptotic approach (see Smith (1959), Feller (1971), Khaniyev and Aksop (2013), Aliyev and Khaniyev (2014), Kesemen, Aliyev and Khaniyev (2013)). When working with inventory model of type (s, S), a variety of distributions can be used for the demand and inter arrival time random variables. For example, in publications of Khaniyev and Aksop (2013) and Khaniyev and Atalay (2010) asymptotic expansion is obtained by assuming that the interference of chance has generalized Beta distribution and triangular distribution respectively. Khaniyev, Kokangul and Aliyev (2013) considered this model with triangular distributed interference of chance and obtained asymptotic expansions for the ergodic moments. Moreover, Aliyev (2016) obtained two term asymptotic expansion for the ergodic distribution function and ergodic moments when the demand distributions are sub exponential with finite variance. A large body of existing literature is based on the assumption that demand distributions are light tailed or have finite variance. However, these assumptions are not fully satisfied in inventory systems, especially when some unexpected fluctuations or extreme values are observed in demand quantities. There are plenty of studies which provide empirical examples for existence of heavy tailed demands (see, for example, Gaffeo, Antonello and Laura (2008), Bimpikis and Markasis (2015),Gaffeo, Antonello and Laura (2008)). The main motivation of this study is the observation of gaps in current literature on consideration of such models with regularly varying random variables with infinite variance. In order to fill a part of this gap, we consider the (s, S) type inventory system with heavy tailed demand distributions and infinite variance. Particularly we consider a special case that the demand distribution $F(x)$ is not arithmetic and $\overline{F} \equiv 1 - F$ is regularly varying, that is, satisfies

$$\frac{\overline{F}(tx)}{\overline{F}(t)} \longrightarrow x^{-\alpha} \qquad \text{as } t \to \infty, 1 < \alpha < 2.$$

It is well known that regularly varying distributions have infinite variance in this case. Differently from the other studies, we obtained our results by using a distinct asymptotic expansion provided by Geluk (1997) for the renewal function. To the best of our knowledge, this is the first study on a semi Markovian inventory model of type (s, S), where the demand random variables have regularly varying distribution with infinite variance.

Regular variation is an elegant concept that come across in many problems related to applied probability. For details about regularly varying random variables, we refer the reader to the textbooks of Seneta (1976), Borokov and Borokov (2008), Bingham, Goldie and Teugels (1987), Resnick (2006). We will give a short summary in Section 2. The remainder of this paper is organized as follows: In Section 3, the ergodicity of the process $X(t)$ is represented. In Section 4, exact and asymptotic results for the ergodic distribution of the process $Y(t)$ is obtained and weak convergence theorem is proved. Here, $Y(t)$ is defined as a standard form of the process $X(t)$ and represented as follows:

$$Y(t) \equiv \frac{2(X(t)-s)}{(S-s)}.$$

2 Preliminaries

Let us give the essential notations and explain this model mathematically before analyzing the main problem.

2.1 Heavy tailed distributions and subclasses

This section covers main definitions and basic results which will appear in the rest of this study. The well-known content is taken from Foss, Korshunov and Zachary (2011), Bingham, Goldie and Teugels (1987).

Definition 2.1. A distribution F on $\mathbb{R}$ is said to be (right) heavy tailed if

$$\int_{-\infty}^{\infty} e^{\lambda x} F(dx) = \infty \qquad \text{for all } \lambda > 0$$

(Foss, Korshunov and Zachary (2011)).

For a comprehensive survey on heavy tailed distributions, see the books by Asmussen (2000), Embrechts, Kluppelberg and Mikosh (1997), Borokov and Borokov (2008), Resnick (2006).

In literature the notion of heavy tails can be used in different senses, for example random variables with subexponential tails, regularly varying tails, regularly varying tails with exponent $\alpha < 1$, infinite variance and infinite mean.

One of the most popular distributions among the heavy tailed distributions which have the widest application area is the distribution with regularly varying tails. Regularly varying distributions behave asymptotically like power functions. To begin the theory of regularly varying distributions and the power law property, we need to introduce regularly varying and slowly varying functions.

Definition 2.2 (Regularly Varying Functions). The positive, measurable function f is called regularly varying at ∞ with index $\alpha \in \mathbb{R}$, if for all $\lambda > 0$

$$\lim_{x \longrightarrow \infty} \frac{f(x\lambda)}{f(x)} = \lambda^{\alpha}.$$

If $\alpha = 0$, then f is called slowly varying function.

The family of regularly varying functions with index α is denoted by $\mathrm{RV}(\alpha)$.

Definition 2.3 (Regularly Varying Random Variables). The nonnegative random variable X and its distribution are called regularly varying with index $\alpha \geq 0$ if the right tail distribution $\overline{F}(x) \in \mathrm{RV}(-\alpha)$.

2.2 Properties of regularly varying functions and distributions

Remark. If $f \in \mathrm{RV}(\alpha)$, then $f(x) = x^{\alpha} L(x)$ where $L(x)$ is a slowly varying function ($L(x) \in \mathrm{RV}(0)$). Hence, we can conclude that any regularly varying distribution can be represented in the following way:

$$P(X > x) = x^{-\alpha} L(x) \qquad \text{where } \alpha > 0 \text{ and } L(x) \in \mathrm{RV}(0).$$

Note that α is the shape parameter and controls the asymptotic behavior of the tail. As α decreases the tail becomes heavier hence $P(X > x)$ decays to zero slower as $x \longrightarrow \infty$, which means extreme values occur more frequently. Karamata theorem says that $L(x)$ can be considered as a constant. This makes easy to work with the general class of regularly varying random variables.

Proposition 2.1 (Bingham, Goldie and Teugels (1987)). *Let L be slowly varying in* $[x_0, \infty)$ *for some* $x_0 \geq 0$. *Then*

1. *for* $\alpha > -1$,

$$\int_{x_0}^{x} t^{\alpha} L(t)\, dt \sim (\alpha + 1)^{-1} x^{\alpha+1} L(x),$$

2. *for* $\alpha < -1$,

$$\int_{x}^{\infty} t^{\alpha} L(t)\, dt \sim -(\alpha + 1)^{-1} x^{\alpha+1} L(x),$$

where by $f(x) \sim g(x)$ *we mean* $\lim_{x \to \infty} f(x)/g(x) = 1$.

Proposition 2.1 is known as Karamata theorem in literature. Proposition 2.2 and Proposition 2.3 allows us to make some operations on regularly varying functions.

Proposition 2.2 (Bingham, Goldie and Teugels (1987)).

1. *If L varies slowly, then*

$$\lim_{x\longrightarrow\infty}\frac{\log(L(x))}{\log(x)}=0.$$

2. *If L varies slowly, so does $(L(x))^{\alpha}$ for every $\alpha\in\mathbb{R}$.*
3. *If L_1, L_2 varies slowly, so do L_1L_2, L_1+L_2. If $L_2(x)\longrightarrow\infty$ as $x\longrightarrow\infty$, then $L_1(L_2(x))$ varies slowly.*
4. *If L varies slowly and $\alpha>0$, then*

$$x^{\alpha}L(x)\longrightarrow\infty,$$

$$x^{-\alpha}L(x)\longrightarrow 0.$$

Proposition 2.3 (Bingham, Goldie and Teugels (1987)).

1. *If $f(x)\in\mathrm{RV}(\alpha)$, then $(f(x))^{p}\in\mathrm{RV}(\alpha p)$ for any $p\in\mathbb{R}$.*
2. *If $f_i\in\mathrm{RV}(\alpha_i)$, $i=1,2$, and $f_2(x)\longrightarrow\infty$ as $x\longrightarrow\infty$, then $f_1(f_2(x))\in\mathrm{RV}(\alpha_1\alpha_2)$.*
3. *If $f_i\in\mathrm{RV}(\alpha_i)$, $i=1,2$, then $f_1(x)+f_2(x)\in\mathrm{RV}(\alpha)$, $\alpha=\max(\alpha_1,\alpha_2)$.*

There are also some other important properties of regularly varying random variables for example self similarity, closure properties etc.. For more detailed information of regularly varying distributions, we refer the reader to Bingham, Goldie and Teugels (1987), Borokov and Borokov (2008), Resnick (2006), Seneta (1976).

2.3 Brief explanation of the process $X(t)$ and essential notations

Suppose that we have a depot and $X(t)$ represents the stock level at this depot at random time t. Assume that z is the initial stock level in a depot at time $t=0$, hence

$$X(0)=X_0=z\in[s,S],\qquad 0<s<S<\infty.$$

Here s is the stock control level and S is the maximum stock level. In addition, suppose that $\{\eta_n\}$, $n\geq 1$ which describes the random amount of demands coming to the system at random times $T_1,T_2,\ldots,T_n,\ldots$. Here $T_n=\sum_{i=1}^{n}\xi_i$, where ξ_n, $n\geq 1$ are inter arrival times between two successive demands η_n, $n\geq 1$. Hence, the stock level $X(t)$ decreases by $\eta_1,\eta_2,\ldots,\eta_n,\ldots$ at random times $T_1,T_2,\ldots,T_n,\ldots$ until the stock level $X(t)$ falls below the control level s, at random time τ_1. In this instance, the stock level changes as follows:

$$X(T_1)\equiv X_1=z-\eta_1,$$

$$X(T_2)\equiv X_2=z-(\eta_1+\eta_2),\ldots,X(T_n)\equiv X_n=z-\sum_{i=1}^{n}\eta_i,$$

where, η_n represents the amount of nth demand, $n = 1, 2, 3, \ldots, \tau_1$ is the first time, that the stock level falls below the control level s. After the stock level falls below s, it is immediately filled up to the level ζ_1, and the first period is completed. Second period starts with a new initial level ζ_1 and continues in a similar manner to the first period.

2.4 Mathematical construction of the process $X(t)$

Let $(\Omega, \Im, P)$ be probability space and $\{(\xi_n, \eta_n, \zeta_n)\}$, $n \geq 1$ be a vector of i.i.d. random variables defined on $(\Omega, \Im, P)$. Here ξ_n and η_n are positive valued random variables. The random variable ζ_n takes values in the interval $[s, S]$. Moreover, ξ_n, η_n and ζ_n are also independent from each other.

Let denote the distributions of ξ_n, η_n and ζ_n by $\Phi(t)$, $F(x)$ and $\pi(z)$ respectively. The distribution functions are defined as follows:

$$\begin{aligned}
\Phi(t) &= P\{\xi_1 \leq t\}, \qquad t \geq 0,\\
F(x) &= P\{\eta_1 \leq x\}, \qquad x \geq 0,\\
\pi(z) &= P\{\zeta_1 \leq z\}, \qquad z \in [s, S].
\end{aligned}$$

ζ_n represents the initial stock level at the beginning of nth period. We assume here that the random variables ζ_n have uniform distribution on the interval $[s, S]$. Moreover, $\{\eta_n\}$, $n \geq 1$ are regularly varying random variables with infinite variance, i.e., $\overline{F} = 1 - F$ is regularly varying with exponent $-\alpha$, $1 < \alpha < 2$.

The renewal sequences $\{T_n\}$ and $\{S_n\}$ defined as:

$$T_0 = S_0 = 0, \qquad T_n = \sum_{i=1}^{n} \xi_i, \qquad S_n = \sum_{i=1}^{n} \eta_i, \qquad n \geq 1.$$

Now define a sequence of integer valued random variables $\{N_n\}, n \geq 0$ as follows:

$$\begin{aligned}
&N_0 = 0, \qquad N_1 = N(z - s) = \inf\{k \geq 1 : z - S_k \leq s\}, \qquad z \in [s, S].\\
&N_{n+1} = \inf\bigl\{k \geq N_n + 1 : \zeta_n - (S_k - S_{N_n}) < s\bigr\}, \qquad n \geq 1.
\end{aligned}$$

Let $\tau_0 = 0$, $\tau_n = T_{N_n} = \sum_{i=1}^{N_n} \xi_i, n \geq 1$, $\nu(t) = \max\{n \geq 0 : T_n \leq t\}, t \geq 0$.

Under these assumptions the desired stochastic process $X(t)$ is constructed as follows:

$$\begin{aligned}
X(t) &= \zeta_n - (\eta_{N_n+1} + \cdots + \eta_{\nu(t)})\\
&= \zeta_n - (S_{\nu(t)} - S_{N_n}), \quad t \in [\tau_n, \tau_{n+1}), n \geq 0.
\end{aligned} \tag{2.1}$$

A realization of the process $X(t)$ is given in Figure 1.

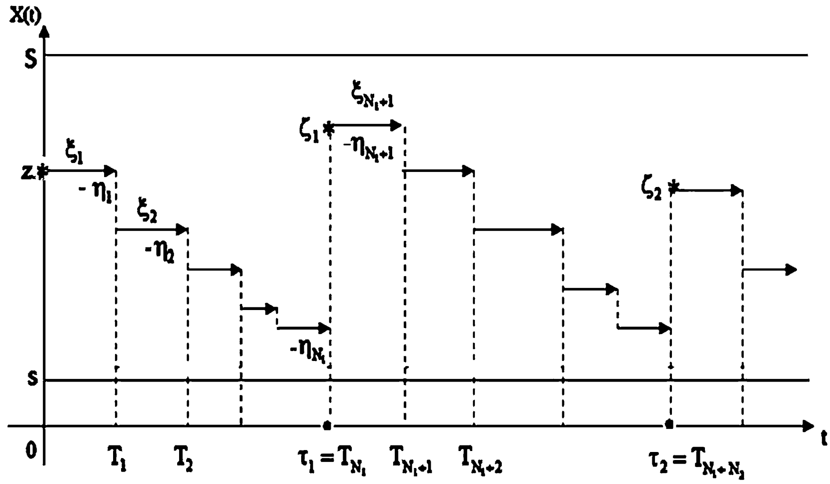

Figure 1 *A Realization of the process $X(t)$.*

3 The ergodicity of the process $X(t)$

Ergodicity of the process $X(t)$ has proven by Khaniyev and Atalay (2010) under some weak conditions. In addition to these conditions, we assume here that the demand random variables $\{\eta_i\}, i \geq 1$ are RV$(-\alpha)$ with $1 < \alpha < 2$. Similarly, we proved the following proposition in order to state the ergodicity of the process $X(t)$.

Proposition 3.1. *Let the initial sequence of random variables $\{(\xi_n, \eta_n, \zeta_n)\}, n \geq 1$ satisfy the following supplementary conditions*:

1. $0 < E(\xi_1) < \infty$.
2. $E(\eta_1) > 0$.
3. $\{\eta_i\}, i \geq 1$ *are non-arithmetic random variables.*
4. *The distribution functions of $\{\eta_i\}$, $i \geq 1$ are regularly varying with index $1 < \alpha < 2$.*
5. *Markov chain $\{\zeta_n\}, n \geq 1$ has uniform distribution on the interval $[s, S]$.*

Then, the process $X(t)$ is ergodic and the following expression is correct with probability 1 for each measurable bounded function $f(x)$, $(f : (s, S) \to \mathbb{R})$

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f\big(X(u)\big)\, du = \frac{\int_s^S \int_s^S f(x)[U_\eta(z-s) - U_\eta(z-x)]\, d\pi(z)\, dx}{\int_s^S U_\eta(z-s)\, d\pi(z)}.$$

Here $U_\eta(x)$ is renewal function generated by sequence $\{\eta_n\}, n \geq 1$.

Proof. The process $X(t)$ is a member of a wide class of processes which is known in the literature as the class of semi-Markov processes with a discrete interference of chance. The general ergodic theorem is provided in the literature for this class (Smith's key renewal theorem) (see Gikhman and Skorohod (1975)). According to the general ergodic theorem we need to prove that the process $X(t)$ meets the following requirements under the conditions of Proposition 3.1.

1. There is a sequence of monotone increasing random times $\{\tau_n\}$, $n = 1, 2, 3, \ldots$ such that $X(\tau_n)$'s forms an ergodic Markov chain.
2. The expected value of the difference between successive stopping times $\{\tau_n\}$, $n = 1, 2, 3, \ldots$ should be finite, that is, $E(\tau_n - \tau_{n-1}) < \infty$.

Let us show that these two assumptions are satisfied.

Assumption 1. In order to define such a Markov chain, we need to provide a sequence of monotone increasing random variables. Consider the sequence of random variables $\{\tau_n\}, n \geq 1$. According to the definition of $\{\tau_n\}, n \geq 1$

$$0 < \tau_1 < \tau_2 < \cdots < \tau_n < \tau_{n+1} < \cdots$$

Moreover τ_n's are the successive times where $X(t)$ falls below the control level s. By the definition, τ_n's are Markov times. According to the mathematical construction of the process $X(t)$, the values of the process at these times are given as $X(\tau_n + 0) = \zeta_n$. Since the sequence of random variables $\{\zeta_n\}, n \geq 1$ are independent and have continuous distribution in the interval $[s, S]$, the sequence $\{X(\tau_n)\} = \zeta_n$ forms an embedded Markov chain. Moreover, the sequence of random variables $\{\zeta_n\}, n \geq 1$ are independent and identically distributed, the embedded Markov chain $\{\zeta_n\}, n \geq 1$ is ergodic with a stationary distribution function

$$\pi(z) = P\{\zeta_1 \leq z\}, \qquad s \leq z \leq S.$$

So the first assumption of the ergodic theorem is satisfied.

Assumption 2. In order to prove second assumption, we need to show that

$$E(\tau_1) < \infty \quad \text{and} \quad E(\tau_n - \tau_{n-1}) < \infty.$$

According to the mathematical construction of the process $X(t)$, $\{\xi_n\}, n \geq 1$ is a sequence of independent and identically distributed random variables. By using Wald identity, we obtain:

$$E\big(\tau_1(z)\big) = E\left(\sum_{i=1}^{N(z)} \xi_i\right) = E(\xi_1)E\big(N(z)\big).$$

According to the assumptions of Proposition 3.1, $E(\xi_1) < \infty$. On the other hand;

$$E\big(N(z)\big) = U(z - s) = 1 + \sum_{n=1}^{\infty} F^{*n}(z - s).$$

Here the function $U(x)$ is renewal function generated by the sequence of random variables $\{\eta_n\}, n \geq 1$. Renewal function $U(x)$ is finite for each finite x (e.g. see Feller (1971)). Therefore, $E(\tau_1(z)) < \infty$.

Now let us prove that

$$E(\tau_n - \tau_{n-1}) < \infty.$$

Since the differences $\tau_1, \tau_2 - \tau_1, \ldots, \tau_n - \tau_{n-1}, \ldots$ have identical distribution, it is sufficient to show that

$$E\big(\tau_1(\zeta)\big) \equiv \int_s^S E\big(\tau_1(z)\big)\, d\pi(z) < \infty.$$

Note that $U(x)$ is renewal function generated by $\eta_n \in \mathrm{RV}(-\alpha)$ with $1 < \alpha < 2$, so with finite mean. Besides being finite for each finite x, the renewal function $U(x)$ is a non decreasing function. Hence for each $z \in [s, S]$, $U(z - s) < U(S - s)$. Therefore, we have:

$$\begin{aligned} E\big(\tau_1(\zeta)\big) &\equiv \int_s^S E\big(\tau_1(z)\big)\, d\pi(z) \\ &= \int_s^S E(\xi_1) E\big(N(z)\big)\, d\pi(z) \\ &= E(\xi_1) \int_s^S E\big(N(z)\big)\, d\pi(z) \\ &= E(\xi_1) \int_s^S U(z-s)\, d\pi(z) \\ &\le E(\xi_1) \int_s^S U(S-s)\, d\pi(z) \\ &\le E(\xi_1) U(S-s). \end{aligned}$$

Since $E(\xi_1) < \infty$ and $U(S - s) < \infty$ then $E(\tau_1(\zeta)) < \infty$. We can conclude that under the conditions of Proposition 3.1, the assumptions of the general ergodic theorem are satisfied. □

A direct result of this Proposition 3.1 is Corollary 3.1 below and obtained by choosing $f(x)$ to be indicator function in Proposition 3.1.

Corollary 3.1. *Let the process $X(t)$ satisfy the conditions of Proposition 3.1. Moreover, let ζ_n, $n \ge 1$ has uniform distribution on the interval $[s, S]$. Then, the ergodic distribution of the process $X(t)$ is given as follows*:

$$Q_X(x) \equiv \lim_{t \longrightarrow \infty} P\big\{X(t) \le x\big\} = 1 - \frac{\int_x^S U_\eta(z-x)\, d\pi(z)}{\int_s^S U_\eta(z-s)\, d\pi(z)}; \qquad x \in [s, S].$$

4 Exact and asymptotic results for the ergodic distribution of the process $Y(t)$

As can be seen in Corollary 3.1, the exact expressions for the ergodic distribution of the process $X(t)$ have a complex structure. The most effective way to deal with this complexity is obtaining asymptotic expansion for $Q_X(x)$ as $S-s \longrightarrow \infty$. In order to obtain an asymptotic expansion for the ergodic distribution of the process, let define a process $Y(t)$ as a standard form of the process $X(t)$ as follows:

$$Y(t) \equiv \frac{X(t)-s}{\beta}, \qquad \beta \equiv \frac{S-s}{2}.$$

Put $Q_Y(\upsilon) = \lim_{t\longrightarrow\infty} P\{Y(t) \leq \upsilon\}$, $\upsilon \in [0,2]$.

Proposition 4.1. *Let the conditions of Proposition 3.1 be satisfied. Moreover, let the random variables ζ_n, $n \geq 1$ have uniform distribution on the interval $[s,S]$. Then, the ergodic distribution function $Q_Y(\upsilon)$ of the process $Y(t)$ is given as follows*:

$$Q_Y(\upsilon) \equiv 1 - \frac{\int_{\beta\upsilon}^{2\beta} U_\eta(x-\beta\upsilon)\,dx}{\int_0^{2\beta} U_\eta(x)\,dx}, \qquad \upsilon \in [0,2].$$

Proof. Recall that, $Q_Y(\upsilon) = \lim_{t\longrightarrow\infty} P\{Y(t) \leq \upsilon\}$, $\upsilon \in [0,2]$. According to the definition of $Y(t)$:

$$Q_Y(\upsilon) = \lim_{t\longrightarrow\infty} P\left\{\frac{X(t)-s}{\beta} \leq \upsilon\right\} = \lim_{t\longrightarrow\infty} P\big\{X(t) \leq \beta\upsilon + s\big\}.$$

In this case;

$$Q_Y(\upsilon) = Q_X(s+\beta\upsilon) = 1 - \frac{\int_{s+\beta\upsilon}^{s+2\beta} U_\eta(z-s-\beta\upsilon)\,d\pi(z)}{\int_s^{s+2\beta} U_\eta(z-s)\,d\pi(z)}; \qquad \upsilon \in [0,2].$$

We assumed that the random variable ζ_n has uniform distribution on the interval $[s,S]$. Hence, the random variable $\tilde{\zeta}_n = \zeta_n - s$ has uniform distribution on the interval $[0,2\beta]$. Define

$$\tilde{\pi}(x) \equiv P\{\tilde{\zeta}_1 \leq x\} = P\{\zeta_1 - s \leq x\} = \pi(s+x).$$

Thus, we have:

$$\begin{aligned} Q_Y(\upsilon) &\equiv 1 - \frac{\int_{\beta\upsilon}^{2\beta} U_\eta(x-\beta\upsilon)\,d\tilde{\pi}(x)}{\int_0^{2\beta} U_\eta(x)\,d\tilde{\pi}(x)} \\ &\equiv 1 - \frac{\frac{1}{2\beta}\int_{\beta\upsilon}^{2\beta} U_\eta(x-\beta\upsilon)\,dx}{\frac{1}{2\beta}\int_0^{2\beta} U_\eta(x)\,dx}. \end{aligned} \tag{4.1}$$

□

The asymptotic expansion for the ergodic distribution function of $Y(t)$ is given trough renewal function. Hence, we need to know the asymptotic expansion for the renewal function generated by the demand random variables. There is sizable literature on the asymptotic expansion for the renewal functions. One of the most well known asymptotic expansion for the renewal function is given by Feller (1971) as follows:

$$U_\eta(x) - \frac{x}{\mu_1} \longrightarrow \frac{\mu_2}{2\mu_1^2}; \qquad \text{as } x \longrightarrow \infty \text{ where } \mu_n = E\left(\eta_1^n\right) < \infty;\ n = 1, 2. \tag{4.2}$$

This expansion is sufficient to obtain the asymptotic expansion for the ergodic distribution of the process $X(t)$ with any light tailed demand, which Khaniyev and Aksop (2013), Khaniyev and Atalay (2010) investigate in their studies. But as we mentioned before, we assumed that the random variables $\{\eta_n\}, n \geq 1$ are heavy tailed. Hence, we need to use a special asymptotic expansion for the renewal function $U_\eta(x)$.

Proposition 4.2 (Geluk (1997)). *Let $F(\cdot)$ be a c.d.f. on $(0, \infty)$ such that $\overline{F}(\cdot) \equiv 1 - F(\cdot)$ regularly varying with exponent $-\alpha$, $1 < \alpha < 2$. Then*

$$\begin{aligned} U_\eta(t) - \frac{t}{\mu_1} - \frac{1}{\mu_1^2} \int_0^t \int_s^\infty \overline{F}(v)\, dv\, ds \\ = O\left(t^4 \left(\overline{F}(t)\right)^2 \overline{F}\left(t^2 \overline{F}(t)\right)\right) \qquad \text{as } t \longrightarrow \infty. \end{aligned} \tag{4.3}$$

Here it is assumed that $\eta_1, \eta_2, \ldots$ is a sequence of i.i.d. real valued positive random variables with d.f. F and $U_\eta(t) = E(N(t))$ is the renewal function associated with $F(t)$.

Remark. In this study, it has been assumed that $\{\eta_n\}, n \geq 1$ is a sequence of regularly varying random variables with exponent $-\alpha$, $1 < \alpha < 2$. Hence, the variance of demands is infinite in this case.

Lemma 4.1. *Let $\{\eta_i\}, i \geq 1$ be a sequence of regularly varying random variables with exponent $-\alpha$, $1 < \alpha < 2$ i.e:*

$$\overline{F}(t) = P\{\eta_1 > t\} = t^{-\alpha} L(t).$$

Then, the renewal function generated by the random variables $\{\eta_i\}, i \geq 1$ is obtained as follows:

$$U_\eta(t) = \frac{t}{\mu_1} + \frac{1}{\mu_1} G(t) + O\left(t^{(\alpha-2)^2} L_1(t)\right), \qquad t \longrightarrow \infty.$$

Where $\mu_1 = E(\eta_1)$. $L_1(t)$ is slowly varying and defined as:

$$L_1(t) = \left(L(t)\right)^2 L\left(t^{2-\alpha} L(t)\right).$$

Note that $1 < \alpha < 2$ *and* $L(t)$ *is the slowly varying function associated with the random variable* η_1. *Moreover,*

$$G(t) = \frac{1}{\mu_1}\int_0^t \int_s^\infty \overline{F}(\upsilon)\,d\upsilon\,ds.$$

Proof. Asymptotic expansion suggested by Geluk (1997) generated by the regularly varying random variables with $1 < \alpha < 2$ is given as follows:.

$$U_\eta(t) = \frac{t}{\mu_1} + \frac{1}{\mu_1}G(t) + O\big(t^4\big(\overline{F}(t)\big)^2\overline{F}\big(t^2\overline{F}(t)\big)\big); \qquad t \longrightarrow \infty.$$

Since $\overline{F}(t) \in \mathrm{RV}(-\alpha)$, then $\overline{F}(t) = t^{-\alpha}L(t)$ where $1 < \alpha < 2$ and $L(t)$ is slowly varying at ∞. Moreover, by Proposition 2.3

$$\overline{F}\big(t^{2-\alpha}L(t)\big) = \big(t^{2-\alpha}\big)^{-\alpha}L\big(t^{2-\alpha}L(t)\big).$$

Hence,

$$\begin{aligned} t^4\big(\overline{F}(t)\big)^2\overline{F}\big(t^2\overline{F}(t)\big) &= t^4t^{-2\alpha}\big(L(t)\big)^2\overline{F}\big(t^2t^{-\alpha}L(t)\big) \\ &= t^{4-2\alpha}\big(L(t)\big)^2\overline{F}\big(t^{2-\alpha}L(t)\big) \\ &= \big(t^{4-2\alpha}\big)\big(t^{\alpha^2-2\alpha}\big)\big(L(t)\big)^2L\big(t^{2-\alpha}L(t)\big) \\ &= t^{(\alpha-2)^2}\big(L(t)\big)^2L\big(t^{2-\alpha}L(t)\big). \end{aligned}$$

Let define $L_1(t) := (L(t))^2L(t^{2-\alpha}L(t))$.

By Proposition 2.2, $(L(t))^2$ is slowly varying function. $t^{2-\alpha}L(t)$ is regularly varying with exponent $(2-\alpha)$ and $L(t)$ is slowly varying.

Moreover, by Proposition 2.2, $t^{2-\alpha}L(t) \longrightarrow \infty$ as $t \longrightarrow \infty$. Hence, by Proposition 2.3, $L(t^{2-\alpha}L(t))$ is also regularly varying with exponent zero, which is a slowly varying function. So by Proposition 2.2, $L_1(t) = (L(t))^2L(t^{2-\alpha}L(t))$ is slowly varying function where $L(t)$ is the slowly varying function associated with random variable η_1. This completes the proof. □

Lemma 4.2. *For any bounded function* $g : \mathbb{R} \longrightarrow \mathbb{R}$ *the following asymptotic relation holds when* $\beta \longrightarrow \infty$:

$$\int_0^{2\beta-\beta\upsilon} x^{(\alpha-2)^2}L_1(x)g(x)\,dx = O\big(\beta^{(\alpha-2)^2+1}L_1(\beta)\big), \qquad \upsilon \in [0,2].$$

Here $L_1(\beta) = (L(\beta))^2L(\beta^{2-\alpha}L(\beta))$ *is slowly varying function and* $L(\beta)$ *is the slowly varying function associated with the random variable* η_1.

Proof. Since $g(x)$ is given as a bounded function, there exists a constant $K > 0$ such that:

$$\begin{aligned}
&\left|\int_0^{2\beta-\beta\upsilon} x^{(\alpha-2)^2} L_1(x) g(x)\,dx\right| \\
&\quad\le \int_0^{2\beta-\beta\upsilon} \left|x^{(\alpha-2)^2} L_1(x) g(x)\right| dx \\
&\quad\le K \int_0^{2\beta-\beta\upsilon} \left|x^{(\alpha-2)^2} L_1(x)\right| dx \\
&\quad= K \int_0^{2\beta-\beta\upsilon} x^{(\alpha-2)^2} L_1(x)\,dx \\
&\quad\sim K \frac{(2\beta-\beta\upsilon)^{(\alpha-2)^2+1}}{(\alpha-2)^2+1} L_1(2\beta-\beta\upsilon), \qquad \upsilon \in [0,2].
\end{aligned}$$

Note that in order to obtain following asymptotic relation, Proposition 2.1 is used.

$$K \int_0^{2\beta-\beta\upsilon} x^{(\alpha-2)^2} L_1(x)\,dx \sim K \frac{(2\beta-\beta\upsilon)^{(\alpha-2)^2+1}}{(\alpha-2)^2+1} L_1(2\beta-\beta\upsilon).$$

Therefore,

$$\int_0^{2\beta-\beta\upsilon} x^{(\alpha-2)^2} L_1(x) g(x)\,dx = O\big(\beta^{(\alpha-2)^2+1} L_1(\beta)\big). \qquad \square$$

Lemma 4.3. *Let define $J(\upsilon)$ as follows*:

$$J(\upsilon) = \frac{1}{2\beta} \int_{\beta\upsilon}^{2\beta} U_\eta(x-\beta\upsilon)\,dx, \qquad \upsilon \in [0,2].$$

Under the conditions of Proposition 3.1 and Proposition 4.2, the following asymptotic expansion holds as $\beta \longrightarrow \infty$:

$$J(\upsilon) = \frac{1}{2\beta}\left[\frac{1}{\mu_1}\frac{(2\beta-\beta\upsilon)^2}{2} + \frac{1}{\mu_1} G_0(2\beta-\beta\upsilon) + O\big(\beta^{(\alpha-2)^2+1} L_1(\beta)\big)\right]. \quad (4.4)$$

Here $L_1(\beta) = (L(\beta))^2 L(\beta^{2-\alpha} L(\beta))$ is slowly varying function, $\upsilon \in [0,2]$, $1 < \alpha < 2$, and

$$G_0(x) = \int_0^x G(t)\,dt = \int_0^x \left[\frac{1}{\mu_1}\int_0^t \int_s^\infty \overline{F}(\upsilon)\,d\upsilon\,ds\right] dt, \qquad x \longrightarrow \infty. \quad (4.5)$$

Proof. It is clear that:

$$\int_0^{2\beta-\beta\upsilon} \frac{t}{\mu_1}\,dt = \frac{1}{\mu_1}\frac{(2\beta-\beta\upsilon)^2}{2}, \qquad \upsilon \in [0,2).$$

Moreover by using the definition of $G_0(x)$ and Proposition 2.1 following asymptotic relation is obtained:

$$
\begin{aligned}
G_0(x) &= \frac{1}{\mu_1}\int_0^x \int_0^t \int_s^\infty \upsilon^{-\alpha} L(\upsilon)\, d\upsilon\, ds\, dt \\
&\sim -\frac{1}{\mu_1}\frac{1}{(1-\alpha)}\int_0^x \int_0^t s^{1-\alpha} L(s)\, ds\, dt \\
&\sim -\frac{1}{\mu_1}\frac{1}{(1-\alpha)}\frac{1}{(2-\alpha)}\int_0^x t^{2-\alpha} L(t)\, dt \\
&\sim -\frac{1}{\mu_1}\frac{1}{(1-\alpha)}\frac{1}{(2-\alpha)}\frac{1}{(3-\alpha)} x^{3-\alpha} L(x),
\end{aligned}
\tag{4.6}
$$

where $L(x)$ is the slowly varying function associated with the random variable η_1. Result is straightforward by using Lemma 4.2. □

Corollary 4.1. *Under the conditions of Lemma 4.3, the following asymptotic expansion holds as $\beta \longrightarrow \infty$:*

$$
J(0) = \frac{1}{2\beta}\left[\frac{1}{\mu_1}\frac{(2\beta)^2}{2} + \frac{G_0(2\beta)}{\mu_1} + O\big(\beta^{(\alpha-2)^2+1} L_1(\beta)\big)\right]. \tag{4.7}
$$

By using Lemma 4.3 and Corollary 4.1 the following main result of this study is obtained.

Theorem 4.1. *Under the conditions of Proposition 3.1 and Lemma 4.1, the following asymptotic expansion is obtained for the ergodic distribution $Q_Y(\upsilon)$, as $\beta \longrightarrow \infty$:*

$$
\begin{aligned}
Q_Y(\upsilon) = {}& \frac{4\upsilon - \upsilon^2}{4} + \frac{1}{2\beta^2}\left[G_0(2\beta)\frac{(\upsilon-2)^2}{4} - G_0(2\beta - \beta\upsilon)\right] \\
&+ O\big(\beta^{(\alpha-2)^2-1} L_1(\beta)\big),
\end{aligned}
$$

where $L_1(\beta) = (L(\beta))^2 L(\beta^{2-\alpha} L(\beta))$ is slowly varying, $L(\beta)$ is the slowly varying function associated with random variable η_1 and $G_0(x)$ is defined by equation (4.5).

Proof. Taking into account that

$$
Q_Y(\upsilon) = 1 - \frac{J(\upsilon)}{J(0)}
$$

for any $\upsilon \in [0,2]$ we have:

$$
\begin{aligned}
Q_Y(\upsilon) &= \frac{\frac{4\beta^2-(2\beta-\beta\upsilon)^2}{2\mu_1} + \frac{G_0(2\beta)-G_0(2\beta-\beta\upsilon)}{\mu_1} + O(\beta^{(\alpha-2)^2+1}L_1(\beta))}{\frac{2\beta^2}{\mu_1}[1+\frac{G_0(2\beta)}{2\beta^2} + O(\beta^{(\alpha-2)^2-1}L_1(\beta))]} \\
&= \frac{\frac{4\beta^2\upsilon-\beta^2\upsilon^2}{4\beta^2} + \frac{G_0(2\beta)-G_0(2\beta-\beta\upsilon)}{2\beta^2} + O(\beta^{(\alpha-2)^2-1}L_1(\beta))}{1+\frac{G_0(2\beta)}{2\beta^2} + O(\beta^{(\alpha-2)^2-1}L_1(\beta))} \\
&= \frac{4\upsilon-\upsilon^2}{4} \\
&\quad + \frac{1}{2\beta^2}\Big[G_0(2\beta) - \Big(\frac{4\upsilon-\upsilon^2}{4}\Big)G_0(2\beta) - G_0(2\beta-\beta\upsilon)\Big] \\
&\quad + O\big(\beta^{(\alpha-2)^2-1}L_1(\beta)\big) \\
&= \frac{4\upsilon-\upsilon^2}{4} + \frac{1}{2\beta^2}\Big[G_0(2\beta)\frac{(\upsilon-2)^2}{4} - G_0(2\beta-\beta\upsilon)\Big] \\
&\quad + O\big(\beta^{(\alpha-2)^2-1}L_1(\beta)\big).
\end{aligned}
\tag{4.8}
$$

□

Theorem 4.1 is the main purpose of this study. Now by using Asymptotic Expansion (4.8), we will obtain weak convergence theorem for the ergodic distribution function $Q_Y(\upsilon)$, as $\beta \longrightarrow \infty$.

4.1 Weak convergence for the ergodic distribution of the process $X(t)$

Theorem 4.2. *Assume that the conditions of Proposition 3.1 and Proposition 4.2 be satisfied. Then, the ergodic distribution $Q_Y(\upsilon)$ of $Y(t)$ converges to $R(\upsilon)$ as $\beta \longrightarrow \infty$, that is, $Q_Y(\upsilon) \longrightarrow R(\upsilon)$, where*

$$R(\upsilon) = \frac{4\upsilon-\upsilon^2}{4}.$$

Proof. By using Proposition 2.1, we obtained the following asymptotic relation:

$$G_0(x) \sim -\frac{1}{\mu_1}\frac{1}{(1-\alpha)(2-\alpha)(3-\alpha)}x^{3-\alpha}L(x) \qquad (x \longrightarrow \infty).$$

Hence,

$$\frac{G_0(2\beta)}{2\beta^2} \sim -\frac{1}{\mu_1}\frac{1}{(1-\alpha)(2-\alpha)(3-\alpha)}\beta^{-(\alpha-1)}L(2\beta) \qquad (\beta \longrightarrow \infty).$$

From Proposition 2.2,

$$\frac{G_0(2\beta)}{2\beta^2} \sim -\frac{1}{\mu_1}\frac{1}{(1-\alpha)(2-\alpha)(3-\alpha)}\beta^{-(\alpha-1)}L(2\beta) \longrightarrow 0 \qquad (\beta \longrightarrow \infty).$$

Moreover,

$$\frac{G_0(2\beta-\beta\upsilon)}{2\beta^2} \sim -\frac{1}{\mu_1}\frac{1}{(1-\alpha)(2-\alpha)(3-\alpha)}(2\beta-\beta\upsilon)^{-(\alpha-1)}L(2\beta-\beta\upsilon).$$

By Proposition 2.2,

$$-\frac{1}{\mu_1}\frac{1}{(1-\alpha)(2-\alpha)(3-\alpha)}(2\beta-\beta\upsilon)^{-(\alpha-1)}L(2\beta-\beta\upsilon) \longrightarrow 0 \qquad (\text{as } \beta \longrightarrow \infty).$$

Hence,

$$\frac{1}{2\beta^2}\left[G_0(2\beta)\frac{(\upsilon-2)^2}{4} - G_0(2\beta-\beta\upsilon)\right] \longrightarrow 0 \qquad (\text{as } \beta \longrightarrow \infty).$$

On the other hand by Proposition 2.2,

$$\beta^{(\alpha-2)^2-1}L_1(\beta) \longrightarrow 0 \qquad \text{as } \beta \longrightarrow \infty.$$

From here we conclude that

$$Q_Y(\upsilon) \longrightarrow R(\upsilon)$$

for all $\upsilon \in [0, 2]$ as $\beta \longrightarrow \infty$. This completes the proof. □

5 Summary and conclusions

In this study, a semi Markovian inventory model of type (s, S) with heavy tailed demand has been considered. Specifically, we obtained our analytical results by assuming that the demand random variables belongs to the regularly varying subclass with finite mean and infinite variance. This model is expressed by means of a renewal reward process with uniform distributed interference of chance. Two term asymptotic expansion for the ergodic distribution is obtained when $\beta \equiv \frac{S-s}{2} \to \infty$. Moreover, weak convergence theorem for the ergodic distribution of the process $Y(t)$ is proved and the exact expression of the limit distribution of $R(\upsilon)$ is derived.

Heavy tailed distributions have recently been used to capture a variety of real world phenomena, such as stock market, economics, earthquake prediction and modeling time delays on the World Wide Web. We hope our results of the use of such distributions in inventory models can be useful reference for future studies. This study can be improved in the future in the following ways: a general formula for the moments of the considered process that covers all regularly varying distributions with infinite variance can be obtained. Moreover by using similar asymptotic approaches the semi Markovian random walk process with heavy tailed distributions can be examined. Applying this approach to different subclasses of heavy tailed distributions is another suggestion for future research.

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