Bektaş *Advances in Difference Equations* (2018) 2018:456
https://doi.org/10.1186/s13662-018-1922-2

Advances in Difference Equations
a SpringerOpen Journal

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Normal curves in n-dimensional Euclidean space

Özcan Bektaş[1*]

*Correspondence:
ozcan.bektas@erdogan.edu.tr
[1]Department of Mathematics, Recep Tayyip Erdogan University, Rize, Turkey

Abstract

In this paper, we give a generalization of normal curves to n-dimensional Euclidean space. Then we obtain a necessary and sufficient condition for a curve to be a normal curve in the n-dimensional Euclidean space. We characterize the relationship between the curvatures for any unit speed curve to be congruent to a normal curve in the n-dimensional Euclidean space. Moreover, the differentiable function $f(s)$ is introduced by using the relationship between the curvatures of any unit speed curve in E^n. Finally, the differential equation characterizing a normal curve can be solved explicitly to determine when the curve is congruent to a normal curve.

MSC: 53A04; 53A07; 34A05

Keywords: Normal curve; Curvatures; Explicit solutions; Position vector

1 Introduction

Rectifying, normal and osculating curves in Euclidean 3-space E^3 are well-known concepts in classical differential geometry of space curves; the position vector always lies in its rectifying plane. The position vector of the rectifying, normal and osculating curves are defined by, respectively,

$$\alpha(s) = \lambda_1(s)T(s) + \mu_1(s)B(s),$$

$$\alpha(s) = \lambda_2(s)N(s) + \mu_2(s)B(s),$$

and

$$\alpha(s) = \lambda_3(s)T(s) + \mu_3(s)N(s),$$

for some differentiable functions λ_1, μ_1, λ_2, μ_2, λ_3 and μ_3 of $s \in I \subset \mathbf{R}$ [1]. Here $T(s)$ is tangent vector field, $N(s)$ is normal vector field and $B(s)$ is a binormal vector field. $\{T(s), N(s), B(s)\}$ is called a Frenet frame field. The rectifying curve in E^3 is given by Chen [1]. After this curve was defined, different viewpoints have been developed concerning the rectifying, normal and osculating curves in differential geometry. Some of the studies in the curve theory have been given as follows.

Ilarslan et al. characterize non-null and null rectifying curves, lying fully in the Minkowski 3-space [2]. Chen and Dillen introduce the idea that the Euclidean rectifying curves are the extremal curves which satisfy the equality case of a general inequality and they find a simple relationship between rectifying curves and the notion of centrodes in mechanics [3]. Furthermore, in [4] and [5], the characterization of a rectifying curve is given in Minkowski 3-space and Euclidean 4-space On the other hand, Cambie et al. examined rectifying curves in n-dimensional Euclidean space [6].

Spacelike, timelike and null normal curves in Minkowski space are investigated in [7, 8] and [9]. The relations between rectifying and normal curves in Minkowski 3-space are obtained in [10]. Normal and rectifying curves are defined in Galilean space in [11] and [12]. The osculating, normal and rectifying binull curves in $\mathbf{R}^6_3$ and $\mathbf{R}^5_2$ are given in [13] and [14]. The concept of a normal curve is given by quaternions in Euclidean space; the semi Euclidean space was addressed by Yıldız and Karakus in [15] and [16]. The rectifying, osculating and normal curves are studied by using octonions in [17].

In this paper, we investigate the properties of the normal curves in n-dimensional Euclidean space by using similar methods as in [6]. We give first some fundamental information about the concept of curves in E^n. Then we characterize normal curves in E^n. We obtain a necessary and sufficient condition for a curve to be a normal curve in n-dimensional Euclidean space. The explicit characterization of the normal curves will be proved.

2 Preliminaries

In this section, we present basic notations on the n-dimensional Euclidean space E^n. Let $\alpha : I \subset \mathbf{R} \to E^n$, $s \in I \to \alpha(s)$ be an arclength parameterized, n times continuously differentiable curve. The curve α is called unit speed curve if $\langle \alpha, \alpha \rangle = 1$, where the function $\langle \cdot, \cdot \rangle$ shows the standard inner product in the n-dimensional Euclidean space E^n given by

$$\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i$$

for each $X = (x_1, x_2, \dots, x_n)$ and $Y = (y_1, y_2, \dots, y_n) \in E^n$. The norm of X is given by $\|X\| = \sqrt{\langle X, X \rangle}$. On the other hand, X is an unit vector, if $\|X\| = 1$. Besides, if the curve α in E^n is an arclength parameterized curve, then $\|\frac{d\alpha}{ds}\| = 1$. The Serret–Frenet formulas for E^n is given by the following equations (see [18]):

$$\begin{aligned}
T'(s) &= \kappa_1(s)N(s),\\
N'(s) &= -\kappa_1(s)T(s) + \kappa_2(s)B_1(s),\\
B_1'(s) &= -\kappa_2(s)N(s) + \kappa_3(s)B_2(s),\\
B_i'(s) &= -\kappa_{i+1}(s)B_{i-1}(s) + \kappa_{i+2}(s)B_{i+1}(s), \quad 2 \le i \le n-3,\\
B_{n-2}'(s) &= -\kappa_{n-1}(s)B_{n-3}(s),
\end{aligned} \tag{1}$$

where κ_1, κ_2, κ_3, …, κ_{n-1} are the curvatures function of the curve and they are positive. For basic information on the theory of curves in differential geometry, see references [19, 20] and [21].

3 Normal curves in n-dimensional Euclidean space

In this section, we generalize some definitions, theorems, and results to normal curves in n-dimensional Euclidean space.

Definition 1 Let $\alpha : I \subset \mathbf{R} \to E^n$, $s \in I \to \alpha(s)$ be an arclength parameterized, n times continuously differentiable curve. α is a normal curve in E^n, if the orthogonal complement of $T(s)$ involve a fixed point. The position vector of a normal curve in E^n is

$$\alpha(s) = \lambda(s)N(s) + \sum_{i=1}^{n-2} \mu_i(s)B_i(s), \tag{2}$$

where we have some differentiable functions λ and μ_i $(1 \le i \le n-2)$ of $s \in I \subset \mathbf{R}$.

Let α be an arc length parameterized normal curve in n-dimensional Euclidean space. By taking the derivative of (2) with respect to s, we obtain the following statement:

$$\alpha'(s) = \lambda'(s)N(s) + \lambda(s)N'(s) + \sum_{i=1}^{n-2} \bigl(\mu_i'(s)B_i(s) + \mu_i(s)B_i'(s)\bigr).$$

By using the Serret–Frenet formulas given with (1) for the curve in the n-dimensional Euclidean space, we get

$$\begin{aligned} T(s) ={}& \bigl(-\lambda(s)\kappa_1(s)\bigr)T(s) + \bigl(\lambda'(s) - \mu_1(s)\kappa_2(s)\bigr)N(s) \\ &+ \sum_{i=1}^{n-3} \bigl(\mu_i'(s) + \mu_{i-1}(s)\kappa_{i+1}(s) - \mu_{i+1}(s)\kappa_{i+2}(s)\bigr)B_i(s) \\ &+ \bigl(\mu_{n-2}'(s) + \mu_{n-3}(s)\kappa_{n-1}(s)\bigr)B_{n-2}(s). \end{aligned}$$

If the mutual coefficients of the vector fields are matched in this last expression, the following statements can be written easily:

$$-\lambda(s)\kappa_1(s) = 1, \tag{3}$$

$$\lambda'(s) - \mu_1(s)\kappa_2(s) = 0, \tag{4}$$

$$\mu_1'(s) + \lambda(s)\kappa_2(s) - \mu_2(s)\kappa_3(s) = 0, \tag{5}$$

$$\mu_i'(s) + \mu_{i-1}(s)\kappa_{i+1}(s) - \mu_{i+1}(s)\kappa_{i+2}(s) = 0, \quad 2 \le i \le n-3, \tag{6}$$

$$\mu_{n-2}'(s) + \mu_{n-3}(s)\kappa_{n-1}(s) = 0. \tag{7}$$

The above statements contain $(n-1)$ curvature functions. The coefficient functions λ and μ_i, $1 \le i \le n-2$, in the position vector of the normal curve can be found with the help of these $(n-1)$ curvature functions. From (3), we can find the following first coefficient function:

$$\lambda(s) = -\frac{1}{\kappa_1(s)}. \tag{8}$$

When the coefficient function (8) is used in Eq. (4), the other coefficient function is as follows:

$$\mu_1(s) = -\frac{1}{\kappa_2(s)}\left(\frac{1}{\kappa_1(s)}\right)'. \tag{9}$$

The coefficient functions are given similarly with the help of the related coefficient functions.

$$\mu_2(s) = -\frac{\kappa_2(s)}{\kappa_3(s)}\left(\frac{1}{\kappa_1(s)}\right) - \frac{1}{\kappa_3(s)}\left(\frac{1}{\kappa_2(s)}\right)'\left(\frac{1}{\kappa_1(s)}\right)' - \frac{1}{\kappa_3(s)}\left(\frac{1}{\kappa_2(s)}\right)\left(\frac{1}{\kappa_1(s)}\right)''. \tag{10}$$

Thus, three coefficient functions are characterized by the use of curvature functions. But there are $(n-4)$ functions to be found. When these functions are calculated, long and complex expressions with curvature functions appear. We will describe some functions as follows to simplify these complex and long expressions. These functions are actually presented in the notation of long and complicated expressions. The first notation function $\psi_{1,0}(s)$ is defined by the following representation:

$$\psi_{1,0}(s) = -\frac{1}{\kappa_2(s)}.$$

In this case, the second coefficient function with the help of this notation function can be written as follows:

$$\mu_1(s) = \psi_{1,0}(s)\left(\frac{1}{\kappa_1(s)}\right)'.$$

Let us define other notation functions along similar lines: $\psi_{2,0}(s)$, $\psi_{2,1}(s)$ and $\psi_{2,2}(s)$ are defined by

$$\psi_{2,0}(s) = -\frac{\kappa_2(s)}{\kappa_3(s)}, \qquad \psi_{2,1}(s) = -\frac{1}{\kappa_3(s)}\left(\frac{1}{\kappa_2(s)}\right)', \qquad \psi_{2,2}(s) = -\frac{1}{\kappa_3(s)}\left(\frac{1}{\kappa_2(s)}\right).$$

In this case, the third coefficient function with the help of this notation functions can be written as follows:

$$\mu_2(s) = \psi_{2,0}(s)\left(\frac{1}{\kappa_1(s)}\right) + \psi_{2,1}(s)\left(\frac{1}{\kappa_1(s)}\right)' + \psi_{2,2}(s)\left(\frac{1}{\kappa_1(s)}\right)''.$$

Similarly, let define the other notation functions: $\psi_{3,0}(s)$, $\psi_{3,1}(s)$, $\psi_{3,2}(s)$ and $\psi_{3,3}(s)$ are introduced by the following statements:

$$\begin{aligned}
\psi_{3,0}(s) &= -\frac{1}{\kappa_4(s)}\left(\frac{\kappa_2(s)}{\kappa_3(s)}\right)', \\
\psi_{3,1}(s) &= -\frac{1}{\kappa_4(s)}\left(\frac{\kappa_2(s)}{\kappa_3(s)}\right) - \frac{1}{\kappa_2(s)}\left(\frac{\kappa_3(s)}{\kappa_4(s)}\right) \\
&\quad - \frac{1}{\kappa_4(s)}\left[\frac{1}{\kappa_3(s)}\left(\frac{1}{\kappa_2(s)}\right)'\right]', \\
\psi_{3,2}(s) &= -\frac{1}{\kappa_4(s)}\left(\frac{1}{\kappa_3(s)}\right)\left(\frac{1}{\kappa_2(s)}\right)' - \frac{1}{\kappa_4(s)}\left[\frac{1}{\kappa_3(s)}\left(\frac{1}{\kappa_2(s)}\right)'\right]', \\
\psi_{3,3}(s) &= -\frac{1}{\kappa_4(s)}\left(\frac{1}{\kappa_3(s)}\right)\left(\frac{1}{\kappa_2(s)}\right).
\end{aligned}$$

In this case, the fourth coefficient function with the help of this notation functions can be written as follows:

$$\mu_3(s) = \psi_{3,0}(s)\left(\frac{1}{\kappa_1(s)}\right) + \psi_{3,1}(s)\left(\frac{1}{\kappa_1(s)}\right)' + \psi_{3,2}(s)\left(\frac{1}{\kappa_1(s)}\right)'' + \psi_{3,3}(s)\left(\frac{1}{\kappa_1(s)}\right)'''.$$

When other notation functions are defined and used, the other coefficient functions can be calculated. Moreover, these functions are generalized, and the following coefficient functions are obtained:

$$\mu_i(s) = \sum_{l=0}^{i} \psi_{i,l}(s)\frac{\partial^l}{\partial s^l}\left(\frac{1}{\kappa_1(s)}\right), \quad 2 \le i \le n-2. \tag{11}$$

Thus we get the following last two coefficient functions for $i = n-3$ and $i = n-2$:

$$\mu_{n-3}(s) = \sum_{l=0}^{n-3} \psi_{n-3,l}(s)\frac{\partial^l}{\partial s^l}\left(\frac{1}{\kappa_1(s)}\right), \tag{12}$$

$$\mu_{n-2}(s) = \sum_{l=0}^{n-2} \psi_{n-2,l}(s)\frac{\partial^l}{\partial s^l}\left(\frac{1}{\kappa_1(s)}\right). \tag{13}$$

Substituting (12) and (13) into (7), we get the following relations:

$$\left(\sum_{l=0}^{n-3} \psi_{n-3,l}(s)\frac{\partial^l}{\partial s^l}\left(\frac{1}{\kappa_1(s)}\right)\right)\kappa_{n-1}(s) = -\frac{\partial}{\partial s}\left(\sum_{l=0}^{n-2} \psi_{n-2,l}(s)\frac{\partial^l}{\partial s^l}\left(\frac{1}{\kappa_1(s)}\right)\right), \tag{14}$$

where

$$\begin{aligned}
&\psi_{1,0}(s) = -\frac{1}{\kappa_2(s)}, \\
&\psi_{2,0}(s) = -\frac{\kappa_2(s)}{\kappa_3(s)}, \qquad \psi_{2,1}(s) = -\frac{1}{\kappa_3(s)}\left(\frac{1}{\kappa_2(s)}\right)', \\
&\psi_{2,2}(s) = -\frac{1}{\kappa_3(s)}\left(\frac{1}{\kappa_2(s)}\right), \\
&\psi_{i,0}(s) = \frac{\psi_{i-2,0}(s)\kappa_i(s) + \psi'_{i-1,0}(s)}{\kappa_{i+1}(s)}, \\
&\psi_{i,k}(s) = \psi_{i-2,l}(s)\kappa_i(s) + \psi'_{i-1,0}(s) + \psi_{i-1,l-1}(s), \\
&\psi_{i,i-2}(s) = \frac{\psi_{i-1,i-3}(s) + \psi'_{i-1,i-2}(s)}{\kappa_{i+1}(s)}, \\
&\psi_{i,i-1}(s) = \frac{\psi_{i-1,i-2}(s)}{\kappa_{i+1}(s)},
\end{aligned} \tag{15}$$

$3 \leq i \leq n-2$, $1 \leq l \leq i-2$. Substituting Eqs. (8), (11) into (2), we get the position vector of the normal curve α; it is given by

$$\alpha(s) = -\frac{1}{\kappa_1(s)}N(s) + \sum_{i=2}^{n-2}\left(\sum_{l=0}^{i}\psi_{i,l}(s)\frac{\partial^l}{\partial s^l}\left(\frac{1}{\kappa_1(s)}\right)\right)B_i(s), \tag{16}$$

where $2 \leq i \leq n-2$. Then we have the following theorem.

Theorem 1 *Let $\alpha(s)$ be a unit speed normal curve in E^n with nonzero curvatures. Then $\alpha(s)$ is congruent to a normal curve in E^n if and only if*

$$\begin{aligned}&\left(\sum_{l=0}^{n-3}\psi_{n-3,l}(s)\frac{\partial^l}{\partial s^l}\left(\frac{1}{\kappa_1(s)}\right)\right)\kappa_{n-1}(s)\\&\quad= -\frac{\partial}{\partial s}\left(\sum_{l=0}^{n-2}\psi_{n-2,l}(s)\frac{\partial^l}{\partial s^l}\left(\frac{1}{\kappa_1(s)}\right)\right)\end{aligned} \tag{17}$$

with $\psi_{i,l}$ introduced by Eq. (15).

Proof Let $\alpha(s)$ be a unit speed normal curve in E^n. If Eqs. (12) and (13) are written in (7), then the statement (17) is obtained.

Conversely, suppose that there is a relationship between curvatures as in (17). Let us define the $Y(s)$ vector using the position curve of the normal curve with the curve $\alpha(s)$ in E^n as follows:

$$Y(s) = \alpha(s) - \lambda(s)N(s) - \sum_{i=1}^{n-2}\mu_i(s)B_i(s). \tag{18}$$

Taking the derivative from both sides of the previous equation with respect to s and by using the Serret–Frenet formulas for the curves in E^n, we obtain the following statements:

$$\begin{aligned}Y'(s) &= \left(\left(\frac{\kappa_2(s)}{\kappa_1(s)}\right) + \left(\frac{1}{\kappa_2(s)}\right)'\left(\frac{1}{\kappa_1(s)}\right)' - \frac{1}{\kappa_2(s)}\left(\frac{1}{\kappa_1(s)}\right)''\right)B_1(s)\\&\quad+ \frac{\partial}{\partial s}\left[\left(\frac{\kappa_3(s)}{\kappa_2(s)}\right)\left(\frac{1}{\kappa_1(s)}\right) + \left(\frac{1}{\kappa_3(s)}\left(\frac{\kappa_2(s)}{\kappa_1(s)}\right)\right)\right]B_2(s)\\&\quad+ \frac{\partial}{\partial s}\left[\frac{1}{\kappa_3(s)}\left(\left(\frac{1}{\kappa_2(s)}\right)'\left(\frac{1}{\kappa_1(s)}\right)' - \frac{1}{\kappa_2(s)}\left(\frac{1}{\kappa_1(s)}\right)''\right)\right]B_2(s)\\&\quad- \left[\left(\frac{\kappa_2(s)}{\kappa_1(s)}\right) + \left(\frac{1}{\kappa_2(s)}\right)'\left(\frac{1}{\kappa_1(s)}\right)' - \frac{1}{\kappa_2(s)}\left(\frac{1}{\kappa_1(s)}\right)''\right]B_1(s)\\&\quad+ \frac{\kappa_4(s)}{\kappa_3(s)}\left[\left(\frac{\kappa_2(s)}{\kappa_1(s)}\right) + \left(\frac{1}{\kappa_2(s)}\right)'\left(\frac{1}{\kappa_1(s)}\right)'\right]B_3(s)\\&\quad- \left(\frac{\kappa_4(s)}{\kappa_3(s)\kappa_2(s)}\left(\frac{1}{\kappa_2(s)}\right)'\right)B_3(s)\\&\quad+ \cdots - \mu_{n-3}(s)\kappa_{n-2}(s)B_{n-4}(s)\\&\quad- \big(\mu'_{n-3}(s) + \mu_{n-2}(s)\kappa_{n-1}(s)\big)B_{n-3}(s)\\&\quad- \big(\mu'_{n-2}(s) + \mu_{n-3}(s)\kappa_{n-1}(s)\big)B_{n-2}(s).\end{aligned}$$

If we compute the above equation, we get

$$\begin{aligned} Y'(s) &= -\left(\sum_{l=0}^{n-3} \psi_{n-3,l}(s)\frac{\partial^l}{\partial s^l}\left(\frac{1}{\kappa_1(s)}\right)\right)\kappa_{n-1}(s) \\ &\quad + \frac{\partial}{\partial s}\left(\sum_{l=0}^{n-2} \psi_{n-2,l}(s)\frac{\partial^l}{\partial s^l}\left(\frac{1}{\kappa_1(s)}\right)\right)B_{n-2}(s). \end{aligned}$$

From (14), we get obtain $Y'(s) = 0$. Therefore $Y(s)$ is constant. Thus $\alpha(s)$ is congruent to a normal curve in E^n. □

Theorem 2 *Let $\alpha(s)$ be a unit speed normal curve in E^n with nonzero curvatures. $\alpha(s)$ is a normal curve if and only if the following statements are satisfied*:

(i) *the principal normal and the first binormal component of the position vector α are given by*

$$\langle \alpha(s), N(s)\rangle = -\frac{1}{\kappa_1(s)}, \qquad \langle \alpha(s), B_1(s)\rangle = -\frac{1}{\kappa_2(s)}\left(\frac{1}{\kappa_1(s)}\right)',$$

(ii) *the first binormal and the second binormal component of the position vector α are given by*

$$\begin{aligned} \langle \alpha(s), B_1(s)\rangle &= -\frac{1}{\kappa_2(s)}\left(\frac{1}{\kappa_1(s)}\right)', \\ \langle \alpha(s), B_2(s)\rangle &= -\frac{1}{\kappa_3(s)}\left[\frac{\kappa_2(s)}{\kappa_1(s)} + \left(\left(\frac{1}{\kappa_2(s)}\right)\left(\frac{1}{\kappa_1(s)}\right)'\right)'\right], \end{aligned}$$

(iii) *the second binormal and the third binormal component of the position vector α are given by*

$$\begin{aligned} \langle \alpha(s), B_2(s)\rangle &= -\frac{1}{\kappa_3(s)}\left[\frac{\kappa_2(s)}{\kappa_1(s)} + \left(\left(\frac{1}{\kappa_2(s)}\right)\left(\frac{1}{\kappa_1(s)}\right)'\right)'\right], \\ \langle \alpha(s), B_3(s)\rangle &= -\frac{1}{\kappa_4(s)}\left(\frac{\kappa_3(s)}{\kappa_2(s)}\left(\frac{1}{\kappa_1(s)}\right)'\right) \\ &\quad - \frac{1}{\kappa_4(s)}\left\{\frac{1}{\kappa_3(s)}\left[\frac{\kappa_2(s)}{\kappa_1(s)} + \left(\left(\frac{1}{\kappa_2(s)}\right)\left(\frac{1}{\kappa_1(s)}\right)'\right)'\right]\right\}', \end{aligned}$$

(iv) *the jth binormal and $(j+1)$th binormal component of the position vector α are given by*

$$\begin{aligned} \langle \alpha(s), B_j(s)\rangle &= \sum_{l=0}^{j} \psi_{j,l}(s)\frac{\partial^l}{\partial s^l}\left(\frac{1}{\kappa_1(s)}\right), \\ \langle \alpha(s), B_{j+1}(s)\rangle &= \sum_{l=0}^{j+1} \psi_{j+1,l}(s)\frac{\partial^l}{\partial s^l}\left(\frac{1}{\kappa_1(s)}\right), \end{aligned}$$

where $3 \le j \le n-3$ and $\psi_{i,l}$ is introduced by Eq. (15).

Proof Let α be an arc length parameterized normal curve in n-dimensional Euclidean space with nonzero curvatures. Taking the inner product of the two sides of (16) with $N(s)$, $B_1(s)$, $B_2(s)$, $B_3(s)$, $B_j(s)$ and $B_{j+1}(s)$, $3 \leq j \leq n-3$, respectively. We get the statements (i), (ii), (iii) and (iv).

Conversely, assume that (i) is given. Differentiating both sides of the first equation of the statement (i) with respect to s and using the Serret–Frenet formulas for the curve in E^n, we get $\langle \alpha(s), T(s)\rangle = 0$. Thus α is a normal curve in E^n. Similarly, if the other statements (ii) and (iii) are valid, we find that α is a normal curve in E^n. Differentiating both sides of the first equation of the statement (iv) with respect to s, then we get

$$\begin{aligned}
&\big\langle \alpha(s), -\kappa_{j+1}(s)B_{j-1}(s) + \kappa_{j+2}(s)B_{j+1}(s)\big\rangle \\
&\quad = \frac{\partial}{\partial s}\left(\sum_{l=0}^{j} \psi_{j,l}(s)\frac{\partial^l}{\partial s^l}\left(\frac{1}{\kappa_1(s)}\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
\big\langle \alpha(s), B_{j-1}(s)\big\rangle = {}& \frac{(\sum_{l=0}^{j+1} \psi_{j+1,l}(s)\frac{\partial^l}{\partial s^l}(\frac{1}{\kappa_1(s)}))\kappa_{j+2}(s)}{\kappa_{j+1}(s)} \\
& - \frac{\frac{\partial}{\partial s}(\sum_{l=0}^{j} \psi_{j,l}(s)\frac{\partial^l}{\partial s^l}(\frac{1}{\kappa_1(s)}))}{\kappa_{j+1}(s)},
\end{aligned}$$

where $3 \leq j \leq n-3$ and $\psi_{i,l}$ introduced by Eq. (15). Here, $j = 3, \ldots, j = n-3$, it can be seen that the curve α is the normal curve in E^n for each case if the defined notation functions are also considered. □

Theorem 3 *Let $\alpha(s)$ be an arc length parameterized curve, lying fully in the n-dimensional Euclidean space with nonzero curvatures. Then α is a normal curve if and only if α lies in some hyperquadrics in E^n.*

Proof First assume that α is a normal curve in E^n. From (17), we get

$$\begin{aligned}
&2\left(\frac{1}{\kappa_1(s)}\right)\left(\frac{1}{\kappa_1(s)}\right)' + 2\left(\frac{1}{\kappa_2(s)}\left(\frac{1}{\kappa_1(s)}\right)'\right)\left(\frac{1}{\kappa_2(s)}\left(\frac{1}{\kappa_1(s)}\right)'\right)' \\
&\quad + 2\left\{\frac{1}{\kappa_3(s)}\left[\frac{\kappa_2(s)}{\kappa_1(s)} + \left(\left(\frac{1}{\kappa_2(s)}\right)\left(\frac{1}{\kappa_1(s)}\right)'\right)'\right]\right\} \\
&\quad \times \left\{\frac{1}{\kappa_3(s)}\left[\frac{\kappa_2(s)}{\kappa_1(s)} + \left(\left(\frac{1}{\kappa_2(s)}\right)\left(\frac{1}{\kappa_1(s)}\right)'\right)'\right]\right\}' \\
&\quad + \cdots + 2\left[\sum_{l=0}^{i} \psi_{i,l}(s)\frac{\partial^l}{\partial s^l}\left(\frac{1}{\kappa_1(s)}\right)\right]\left[\sum_{l=0}^{i} \psi_{i,l}(s)\frac{\partial^l}{\partial s^l}\left(\frac{1}{\kappa_1(s)}\right)\right]' \\
&\quad + \cdots + 2\left[\sum_{l=0}^{n-2} \psi_{n-2,l}(s)\frac{\partial^l}{\partial s^l}\left(\frac{1}{\kappa_1(s)}\right)\right]\left[\sum_{l=0}^{n-2} \psi_{n-2,l}(s)\frac{\partial^l}{\partial s^l}\left(\frac{1}{\kappa_1(s)}\right)\right]' \\
&= 0.
\end{aligned}$$

On the other hand, the above statement is a differential version of the following statement:

$$
\begin{aligned}
&\left(\frac{1}{\kappa_1(s)}\right)^2+\left(\frac{1}{\kappa_2(s)}\left(\frac{1}{\kappa_1(s)}\right)'\right)^2\\
&\quad+\left\{\frac{1}{\kappa_3(s)}\left[\frac{\kappa_2(s)}{\kappa_1(s)}+\left(\left(\frac{1}{\kappa_2(s)}\right)\left(\frac{1}{\kappa_1(s)}\right)'\right)'\right]\right\}^2\\
&\quad+\cdots+\left[\sum_{l=0}^{i}\psi_{i,l}(s)\frac{\partial^l}{\partial s^l}\left(\frac{1}{\kappa_1(s)}\right)\right]^2\\
&\quad+\cdots+\left[\sum_{l=0}^{n-2}\psi_{n-2,l}(s)\frac{\partial^l}{\partial s^l}\left(\frac{1}{\kappa_1(s)}\right)\right]^2\\
&=r,\quad r\in\mathbf{R}.
\end{aligned}
$$

Substituting (8)–(13) into (18), and taking the inner product of the both sides of (18) with $\alpha(s)-Y(s)$, then we get the following relations:

$$\bigl\langle\alpha(s)-Y(s),\alpha(s)-Y(s)\bigr\rangle=r.$$

Thus, the desired expression is proved.

Conversely, if α lies in some hyperquadrics in E^n, then $\langle\alpha(s)-Y(s),\alpha(s)-Y(s)\rangle=r$, where $Y(s)\in E^n$ is a constant vector. Taking into account the derivative of the previous equation with respect to s, we find that $\langle\alpha(s)-Y(s),T(s)\rangle=0$. Hence α is a normal curve in E^n. □

The following lemma can be given as a result of Theorem 1.

Lemma 1 *Let $\alpha(s)$ be an arc length parameterized curve, lying fully in E^n, with non-null vector fields $N(s),B_1(s),B_2(s),\dots,B_{n-3}(s)$ and $B_{n-2}(s)$, then we have congruence to a normal curve if and only if there exists a differentiable function $f(s)$ such that*

$$
\begin{aligned}
f(s)\kappa_{n-1}(s)&=\sum_{l=0}^{n-2}\psi_{n-2,l}(s)\kappa_{n-1}(s)\frac{\partial^l}{\partial s^l}\left(\frac{1}{\kappa_1(s)}\right),\\
f'(s)&=-\left(\sum_{l=0}^{n-3}\psi_{n-3,l}(s)\frac{\partial^l}{\partial s^l}\left(\frac{1}{\kappa_1(s)}\right)\right)\kappa_{n-1}(s).
\end{aligned}
\tag{19}
$$

If we apply a similar procedure to Refs. [22, 23] and [24] together with Lemma 1, then we obtain the following theorem when the curves in E^n are normal curves.

Theorem 4 *Let $\alpha(s)$ be an arc length parameterized curve in n-dimensional Euclidean space with nonzero curvatures. Then α is congruent to a normal curve if and only if there exist constant a_0, $b_0\in\mathbf{R}$ such that*

$$
\begin{aligned}
\frac{f'(s)}{\kappa_{n-1}(s)}&=\left\{\int\left[\frac{f''(s)}{\kappa_{n-1}(s)}-\frac{f'(s)\kappa'_{n-1}(s)}{\kappa^2_{n-1}(s)}+f(s)\kappa_{n-1}(s)\right]\cos\theta\,ds-a_0\right\}\cos\theta\\
&\quad+\left\{\int\left[\frac{f''(s)}{\kappa_{n-1}(s)}-\frac{f'(s)\kappa'_{n-1}(s)}{\kappa^2_{n-1}(s)}+f(s)\kappa_{n-1}(s)\right]\sin\theta\,ds-b_0\right\}\sin\theta,
\end{aligned}
\tag{20}
$$

where $\theta(s)=\int_0^s\kappa_{n-1}(s)\,ds$.

Proof Let $\alpha(s)$ be congruent to a normal curve. From Lemma 1, there exists a differentiable function $f(s)$ which can be given with (19) and $f'(s)$ shows that $f'(s) + (\sum_{l=0}^{n-3} \psi_{n-3,l}(s)\frac{\partial^l}{\partial s^l}(\frac{1}{\kappa_1(s)}))\kappa_{n-1}(s) = 0$. Let us describe differentiable functions $\theta(s)$, $a(s)$ and $b(s)$

$$\theta(s) = \int_0^s \kappa_{n-1}(s)\,ds, \tag{21}$$

$$\begin{aligned} a(s) = {} & -\frac{f'(s)}{\kappa_{n-1}(s)}\cos\theta - f(s)\sin\theta \\ & + \int \left[\frac{f''(s)}{\kappa_{n-1}(s)} - \frac{f'(s)\kappa'_{n-1}(s)}{\kappa^2_{n-1}(s)} + f(s)\kappa_{n-1}(s)\right]\cos\theta\,ds, \end{aligned} \tag{22}$$

$$\begin{aligned} b(s) = {} & -\frac{f'(s)}{\kappa_{n-1}(s)}\sin\theta + f(s)\cos\theta \\ & + \int \left[\frac{f''(s)}{\kappa_{n-1}(s)} - \frac{f'(s)\kappa'_{n-1}(s)}{\kappa^2_{n-1}(s)} + f(s)\kappa_{n-1}(s)\right]\sin\theta\,ds. \end{aligned} \tag{23}$$

If we differentiate Eqs. (22) and (23) according to s and consider (19) and (21), then we get $a'(s) = 0$ and $b'(s) = 0$. Thus, $a(s) = a_0$ and $b(s) = b_0 \in \mathbf{R}$. Equations (22) and (23) are multiplied by $\cos\theta$ and $\sin\theta$, respectively, and if the obtained statements are collected, then we get (20).

On the contrary, there are $a_0, b_0 \in \mathbf{R}$ that lead to (20). By the derivative of (20) according to s, we obtain

$$\begin{aligned} & \left(\frac{f'(s)}{\kappa_{n-1}(s)}\right)' \\ & \quad = \frac{f''(s)}{\kappa_{n-1}(s)} - \frac{f'(s)\kappa'_{n-1}(s)}{\kappa^2_{n-1}(s)} + f(s)\kappa_{n-1}(s) \\ & \qquad + \left\{\int \left[\frac{f''(s)}{\kappa_{n-1}(s)} - \frac{f'(s)\kappa'_{n-1}(s)}{\kappa^2_{n-1}(s)} + f(s)\kappa_{n-1}(s)\right]\cos\theta\,ds + a_0\right\}\sin\theta\kappa_{n-1}(s) \\ & \qquad + \left\{\int \left[\frac{f''(s)}{\kappa_{n-1}(s)} - \frac{f'(s)\kappa'_{n-1}(s)}{\kappa^2_{n-1}(s)} + f(s)\kappa_{n-1}(s)\right]\sin\theta\,ds - b_0\right\}\cos\theta\kappa_{n-1}(s). \end{aligned}$$

The differentiable function $f(s)$ is defined by the following statement:

$$f(s) = \frac{1}{\kappa_{n-1}(s)}\left[\sum_{l=0}^{n-2} \psi_{n-2,l}(s)\kappa_{n-1}(s)\frac{\partial^l}{\partial s^l}\left(\frac{1}{\kappa_1(s)}\right)\right].$$

Thus, we get

$$\begin{aligned} & f(s)\kappa_{n-1}(s) \\ & \quad = \left\{\int \left[\frac{f''(s)}{\kappa_{n-1}(s)} - \frac{f'(s)\kappa'_{n-1}(s)}{\kappa^2_{n-1}(s)} + f(s)\kappa_{n-1}(s)\right]\cos\theta\,ds + a_0\right\}\sin\theta \\ & \qquad + \left\{\int \left[\frac{f''(s)}{\kappa_{n-1}(s)} - \frac{f'(s)\kappa'_{n-1}(s)}{\kappa^2_{n-1}(s)} + f(s)\kappa_{n-1}(s)\right]\sin\theta\,ds - b_0\right\}\cos\theta. \end{aligned}$$

Finally, we obtain $f'(s) = -(\sum_{l=0}^{n-3} \psi_{n-3,l}(s) \frac{\partial^l}{\partial s^l}(\frac{1}{\kappa_1(s)}))\kappa_{n-1}(s)$. Lemma 1 implies that α is congruent to a normal curve. □

4 Conclusion

In this paper, we present normal curves and some of their properties in n-dimensional Euclidean space E^n. The necessary and sufficient conditions for a unit speed curve to be congruent to a normal curve in E^n have been characterized in terms of its curvatures and the related differentiable function has been given. The results of this study can also be investigated in different spaces.

Acknowledgements
The author would like to thank the reviewers for their valuable and constructive comments.

Funding
The author received no funding for this work.

Competing interests
The author declares that he has no competing interests.

Authors' contributions
The author read and approved the final manuscript.

Publisher's Note

Received: 14 August 2018 Accepted: 4 December 2018 Published online: 11 December 2018

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