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4 authors, including:

Arzu Bilgin
Recep Tayyip Erdoğan Üniversitesi

Mustafa R. S. Kulenović
University of Rhode Island

Esmir Pilav
University of Sarajevo

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Global Dynamics of a Cooperative Discrete System in the Plane

Arzu Bilgin
Department of Mathematics,
Recep Tayyip Erdogan University, Rize, Turkey
arzu.bilgin@erdogan.edu.tr

Ann Brett
Department of Mathematics, Johnson and Wales University,
Providence, Rhode Island 02903, USA
ambrett@verizon.net

Mustafa R. S. Kulenović
Department of Mathematics, University of Rhode Island,
Kingston, Rhode Island 02881, USA
mkulenovic@uri.edu

Esmir Pilav
Department of Mathematics, University of Sarajevo,
Sarajevo 71000, Bosnia and Herzegovina
esmir.pilav@pmf.unsa.ba

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In this paper, we consider the cooperative system

$$x_{n+1} = ax_n + \frac{by_n^2}{1+y_n^2}, \quad y_{n+1} = \frac{cx_n^2}{1+x_n^2} + dy_n, \quad n = 0, 1, \ldots,$$

where all parameters a, b, c, d are positive numbers and the initial conditions x_0, y_0 are non-negative numbers. We describe the global dynamics of this system in a number of cases. An interesting feature of this system is that it exhibits a coexistence of locally stable equilibrium and locally stable periodic solutions as well as the Allee effect.

Keywords: Allee effect; basin; cooperative map; invariant manifold; stable manifold.

1. Introduction and Preliminaries

In this paper, we consider the cooperative system

$$\begin{aligned} x_{n+1} &= ax_n + \frac{by_n^2}{1+y_n^2}, \\ y_{n+1} &= \frac{cx_n^2}{1+x_n^2} + dy_n, \end{aligned} \quad n = 0, 1, \ldots, \tag{1}$$

where all parameters a, b, c, d are positive numbers and the initial conditions x_0, y_0 are non-negative numbers. In view of the following preliminary result we will restrict our attention to the case a, $d \in (0, 1)$.

Lemma 1. *Consider system (1).*

(i) *If* $a \geq 1$ *then* $\lim_{n\to\infty} x_n = \infty$ *and* $\lim_{n\to\infty} y_n = \infty$ *if* $d \geq 1$ *and* $\lim_{n\to\infty} y_n = \frac{c}{1-d}$, *if* $d < 1$.
(ii) *If* $d \geq 1$ *then* $\lim_{n\to\infty} y_n = \infty$ *and* $\lim_{n\to\infty} x_n = \infty$ *if* $a \geq 1$ *and* $\lim_{n\to\infty} x_n = \frac{b}{1-a}$, *if* $a < 1$.

Proof

(i) If $a \geq 1$ then the first equation of system (1) implies $x_{n+1} > ax_n \geq x_n$, which shows that $\{x_n\}_{n=1}^{\infty}$ is an increasing sequence and because there is no positive equilibrium in this case we have that $\lim_{n\to\infty} x_n = \infty$. From the perspective of theorem on difference inequalities, see [Lakshmikantham & Trigiante, 2002], $\{y_n\}_{n=1}^{\infty}$ is converging to the asymptotic solution of the limiting equation

$$y_{n+1} = c + dy_n, \quad n = 1, 2, \dots$$

which completes the proof in this case.

(ii) The proof in this case is similar to the proof of Case 1 and is omitted. ∎

System (1) is a cooperative system with interspecific cooperation coefficients $\frac{by_n^2}{1+y_n^2}$ and $\frac{cx_n^2}{1+x_n^2}$ that are Beverton–Holt functions. The related system

$$\begin{aligned} x_{n+1} &= ax_n + \frac{by_n}{1+y_n}, \\ y_{n+1} &= \frac{cx_n}{1+x_n} + dy_n, \end{aligned} \qquad n = 0, 1, \dots, \tag{2}$$

where all parameters a, b, c, d are positive numbers and the initial conditions x_0, y_0 are non-negative numbers exhibiting a simple exchange of stability bifurcation for the critical value of the coefficients $bc - (1-a)(1-d)$, see [Bilgin & Kulenović, 2017]. More precisely, when $a, d \in (0, 1)$, the zero equilibrium of system (2) is globally asymptotically stable if $bc - (1-a)(1-d) \leq 0$ and the positive equilibrium is globally asymptotically stable if $bc - (1-a)(1-d) > 0$. The introduction of the Beverton–Holt of the quadratic terms of the Beverton–Holt sigmoid function changes the global behavior of the system by introducing the period-two solutions which, under certain conditions can be locally stable.

From a modeling point of view system (1) simultaneously exhibits the Allee effect, a globally stable positive equilibrium solution(s) relative to its basin of attraction, and a globally stable period-two solution(s) relative to its basin of attraction.

System (1) may have very complicated dynamics but in the cases when it has one, two or three period-two solutions we can determine its global dynamics. We were able to find the upper bound for the number of period-two solutions which is 28, but we were not able to exclude the existence of the periodic solutions of other periods.

One can show that system (1) does not satisfy either the $(O+)$ or the $(O-)$ condition which means that we cannot conclude that all solutions are converging to an equilibrium solution or to a period-two solution. Thus, there is a possibility that system (1) may have periodic solutions of different periods.

Since system (1) is strictly cooperative, Sharkovskii's ordering holds for periodic solutions [Wang & Jiang, 2001] and so for instance the existence of a period-three solution would imply the existence of periodic solutions of all other periods.

Some monotone systems that were considered in [Brett & Kulenović, 2014, 2015; Hadžiabdić *et al.*, 2014] satisfy the $(O+)$ condition and so have simpler dynamics. Examples of monotone systems that exhibit chaos were given in [Smith, 1998; Wang & Jiang, 2001]. The results that we obtain in this paper imply that when system (1) has period-two solutions which are locally asymptotically stable, saddle points or nonhyperbolic points of the stable type, then every solution converges to either an equilibrium solution or to period-two solutions. So the necessary condition for system (1) to have a periodic solution of period three or higher is that all period-two solutions are either repellers or nonhyperbolic points of the unstable type.

System (1) is an example of a strongly cooperative system with coexisting locally stable equilibria and locally stable period-two solutions. A first such example was given in [Dancer & Hess, 1991] where the *arctan* function was used and so topological rather than computational proofs were provided. The advantage of system (1) is that it is an algebraic example, which allows the computational proofs and algebraic conditions for the existence of one, two or three equilibrium solutions to be clearly stated. We believe that our main results for system (1) can be extended to the general cooperative system

$$\begin{aligned} x_{n+1} &= f(x_n, y_n), \\ y_{n+1} &= g(x_n, y_n), \end{aligned} \qquad n = 0, 1, \dots, \tag{3}$$

where f, g are increasing functions in their variables, with the same configuration and local character of equilibrium solutions and period-two solutions. Finally, system (1) is a feasible model in population dynamics since most transition functions are either linear or Beverton–Holt sigmoid functions.

The principal tool that we will use in our proofs are two results on basins of attraction of monotone maps which we provide in the rest of this section, and results on the existence of stable and unstable manifolds for monotone maps in the plane from [Kulenović & Merino, 2006, 2009, 2010]. In the rest of this section we list basic notation which will be used throughout the paper.

Let $\preceq$ be a partial order on $\mathbb{R}^n$ with nonnegative cone P. For $x, y \in \mathbb{R}^n$ the *order interval* $[\![x, y]\!]$ is the set of all z such that $x \preceq z \preceq y$. We say $x \prec y$ if $x \preceq y$ and $x \neq y$, and $x \ll y$ if $y - x \in \text{int}(P)$. A map T on a subset of $\mathbb{R}^n$ is *order preserving* if $T(x) \preceq T(y)$ whenever $x \prec y$, *strictly order preserving* if $T(x) \prec T(y)$ whenever $x \prec y$, and *strongly order preserving* if $T(x) \ll T(y)$ whenever $x \prec y$.

Let $T : R \to R$ be a map with a fixed point $\overline{x}$ and let R' be an invariant subset of R that contains $\overline{x}$. We say that $\overline{x}$ is stable (asymptotically stable) relative to R' if $\overline{x}$ is a stable (asymptotically stable) fixed point of the restriction of T to R'.

Throughout this paper we shall use the *North-East ordering* (NE) for which the positive cone is the first quadrant, i.e. this partial ordering is defined by $(x_1, y_1) \preceq_{ne} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ and the South-East (SE) ordering defined as $(x_1, y_1) \preceq_{se} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \geq y_2$.

A map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ which is monotone with respect to the North-East (NE) ordering is called *cooperative* and a map monotone with respect to the South-East (SE) ordering is called *competitive*. A map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ of which the second iterate T^2 is monotone with respect to the North-East (resp., South-East) ordering is called *anti-cooperative* (resp., *anti-competitive*), see [Kalabušić *et al.*, 2013].

If T is a differentiable map on a nonempty set $\mathcal{R}$, a sufficient condition for T to be strongly monotone with respect to the NE ordering is that the Jacobian matrix at all points x has the sign configuration

$$\text{sign}(J_T(\mathbf{x})) = \begin{bmatrix} + & + \\ + & + \end{bmatrix}, \tag{4}$$

provided that $\mathcal{R}$ is open and convex. An equilibrium $\mathbf{x}$ of cooperative or competitive system (3) is said to be nonhyperbolic of stable (resp., unstable) type if one of the eigenvalues of the Jacobian matrix evaluated at $\mathbf{x}$ is by absolute value 1 and the second one is by absolute value less (resp., more) than 1.

For $(x_1, x_2) \in \mathbb{R}^2$, define $Q_\ell(x_1, x_2)$ for $\ell = 1, 2, 3, 4$ to be the usual four quadrants based at x and numbered in a counterclockwise direction, for example, $Q_1(x_1, x_2) = \{y = (y_1, y_2) \in R^2 : x_1 \leq y_1, x_2 \leq y_2\}$. The basin of attraction of a fixed point $(\overline{x}, \overline{y})$ of a map T, denoted as $\mathcal{B}((\overline{x}, \overline{y}))$, is defined as the set of all initial points (x_0, y_0) for which the sequence of iterates $T^n((x_0, y_0))$ converges to $(\overline{x}, \overline{y})$. Similarly, we define a basin of attraction of a periodic point of period p.

Let T be a cooperative map defined on $\mathcal{R} \subset \mathbb{R}^2$. The map T is said to satisfy the property $(O+)$ (resp., $(O-)$)

$$\text{if } x, y \in \mathcal{R} \text{ are such that } T(x) \preceq_{se} T(y), \quad \text{then } x \preceq_{se} y, \tag{O+}$$

respectively,

$$\text{if } x, y \in \mathcal{R} \text{ are such that } T(x) \preceq_{se} T(y), \quad \text{then } y \preceq_{se} x. \tag{O−}$$

The well-known deMottoni–Schiaffino theorem, see [Kulenović & Merino, 2009; Smith, 1998], claims that in this case for each $x \in \mathcal{R}$, the sequence $\{T^n(x)\}$ (resp., $\{T^{2n}(x)\}$) is eventually coordinatewise monotonic. Consequently, every bounded sequence $\{T^n(x)\}$ (resp., $\{T^{2n}(x)\}$) converges to a fixed point of T or to a point on the boundary of $\mathcal{R}$.

The next result in [Kulenović & Merino, 2009] is stated for order preserving maps on $\mathbb{R}^n$. See [Hess, 1991] for a more general version valid in ordered Banach spaces.

Theorem 1. *For a nonempty set $R \subset \mathbb{R}^n$ and $\preceq$ a partial order on $\mathbb{R}^n$, let $T : R \to R$ be an order preserving map, and let $a, b \in R$ be such that $a \prec b$ and $[\![a, b]\!] \subset R$. If $a \preceq T(a)$ and $T(b) \preceq b$, then $[\![a, b]\!]$ is an invariant set and*

(i) *There exists a fixed point of T in $[\![a, b]\!]$.*
(ii) *If T is strongly order preserving, then there exists a fixed point in $[\![a, b]\!]$ which is stable relative to $[\![a, b]\!]$.*
(iii) *If there is only one fixed point in $[\![a, b]\!]$, then it is a global attractor in $[\![a, b]\!]$ and therefore asymptotically stable relative to $[\![a, b]\!]$.*

The following result is a direct consequence of the Trichotomy Theorem of Dancer and Hess, see [Dancer & Hess, 1991; Hess, 1991; Kulenović & Merino, 2009], and is helpful for determining the basins of attraction of the equilibrium points.

Corollary 1.1. *If the non-negative cone of a partial ordering $\preceq$ is a generalized quadrant in $\mathbb{R}^n$, and if T has no fixed points in $[\![u_1, u_2]\!]$ other than u_1 and u_2, then the interior of $[\![u_1, u_2]\!]$ is either a subset of the basin of attraction of u_1 or a subset of the basin of attraction of u_2.*

2. Local Stability Analysis of Equilibrium Solutions

First, we discuss the existence of the equilibrium solutions.

The equilibrium points of the system (1) satisfy the following system of equations:

$$\bar{x} = a\bar{x} + \frac{b\bar{y}^2}{1+\bar{y}^2}, \quad \bar{y} = \frac{c\bar{x}^2}{1+\bar{x}^2} + d\bar{y}. \tag{5}$$

It follows immediately that the zero equilibrium is a solution of (5).

Geometrically, solutions of (5) are intersections of two orthogonal rational curves:

$$\bar{x} = \frac{b\bar{y}^2}{(1-a)(1+\bar{y}^2)}, \quad \bar{y} = \frac{c\bar{x}^2}{(1-d)(1+\bar{x}^2)}. \tag{6}$$

See Fig. 1. It follows from (6) that necessary conditions for non-negative equilibrium points are $a < 1$ and $d < 1$.

Rearranging (6), we have the following equations,

$$\begin{aligned} &(\mathcal{E}_1)\colon\ a\bar{x} + a\bar{x}\bar{y}^2 + b\bar{y}^2 - \bar{x} - \bar{x}\bar{y}^2 = 0,\\ &(\mathcal{E}_2)\colon\ c\bar{x}^2 + d\bar{y} + d\bar{x}^2\bar{y} - \bar{y} - \bar{x}\bar{y}^2 = 0. \end{aligned} \tag{7}$$

From (7) one can see that all positive solutions of system (1) satisfy:

$$\begin{aligned}(1-a)((d-1)^2+c^2)\bar{x}^5 - bc^2\bar{x}^4&\\ + 2(d-1)^2(1-a)\bar{x}^3 + (1-a)(d-1)^2x = 0&\end{aligned} \tag{8}$$

and

$$\begin{aligned}(d-1)((1-a)^2+b^2)\bar{y}^5 + b^2c\bar{y}^4&\\ + 2(a-1)^2(d-1)\bar{y}^3 + (a-1)^2(d-1)\bar{y} = 0.&\end{aligned} \tag{9}$$

The left-hand side of (8) is a quintic polynomial. Since $a < 1$ and $d < 1$, the polynomial has coefficients which have two changes of sign. Consequently, by Descartes' rule of sign, Eq. (8) has either zero, one, or two roots.

Consequently, system (1) has the zero equilibrium always and either zero, one or two positive equilibrium solutions.

These equilibrium solutions will be denoted $E_0(0,0)$, $E(\bar{x},\bar{y})$, $E_{SW}(\bar{x},\bar{y})$ and $E_{NE}(\bar{x},\bar{y})$.

Lemma 2. *Assume that $a < 1$ and $d < 1$. Let*

$$\begin{aligned}\Delta = 27(a-1)b^4c^8&\\ - 32(a-1)^3c^6(d-1)^2(8(a-1)^2+9b^2)&\\ - 256(a-1)^3c^4(d-1)^4((a-1)^2+b^2).&\end{aligned}$$

Consider the equation

$$\begin{aligned}(1-a)(c^2+(d-1)^2)\bar{x}^4 - bc^2\bar{x}^3&\\ + 2(1-a)(d-1)^2\bar{x}^2 + (1-a)(d-1)^2 = 0.&\end{aligned} \tag{10}$$

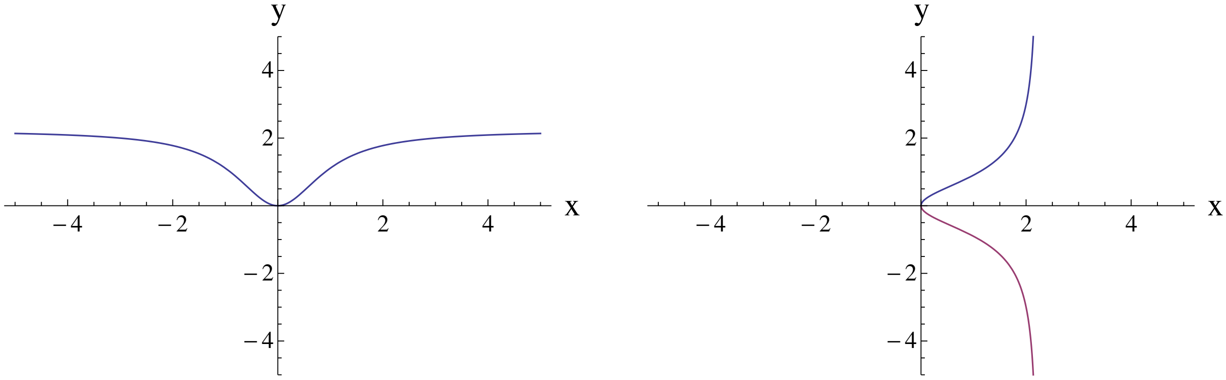


Fig. 1. The curves of the equilibrium points for $b = c = 1$, $a = d = 1/2$.

Then the following holds:

(a) *All real roots of Eq. (10) are positive numbers. Furthermore, if* $(\overline{x}, \overline{y})$ *is a real solution of system (5) then* $\overline{x} \geq 0$ *and* $\overline{y} \geq 0$.
(b) *If* $\Delta > 0$, *then Eq. (10) has zero real roots and two pairs of distinct conjugate imaginary roots. Consequently, system (1) has one equilibrium point* $E_0(0,0)$.
(c) *If* $\Delta < 0$, *then Eq. (10) has two distinct real roots and one pair of conjugate imaginary roots. Consequently, system (1) has three equilibrium points* $E_0(0,0)$, $E_{SW}(\overline{x}, \overline{y})$ *and* $E_{NE}(\overline{x}, \overline{y})$.
(d) *If* $\Delta = 0$, *then Eq. (10) has one pair of conjugate imaginary roots and one real root of multiplicity two. Consequently, system (1) has two equilibrium points* $E_0(0,0)$, *and* $E(\overline{x}, \overline{y})$.

Proof. The proof of (a) follows from Descartes' rule of signs. Let

$$\begin{aligned} f(x) &= (1-a)(c^2+(d-1)^2)x^4 - bc^2x^3 \\ &\quad + 2(-a-1)(d-1)^2x^2 + (1-a)(d-1)^2. \end{aligned}$$

The following matrix, called the discrimination matrix of $f(x)$ and $f'(x)$ in [Yang *et al.*, 1996], is actually the Sylvester matrix of $f(x)$ and $f'(x)$ with some permuted rows, as given by

$$\mathrm{Discr}(\tilde{f}) = \begin{pmatrix} a_4 & a_3 & a_2 & 0 & a_0 & 0 & 0 & 0 \\ 0 & 4a_4 & 3a_3 & 2a_2 & 0 & 0 & 0 & 0 \\ 0 & a_4 & a_3 & a_2 & 0 & a_0 & 0 & 0 \\ 0 & 0 & 4a_4 & 3a_3 & 2a_2 & 0 & 0 & 0 \\ 0 & 0 & a_4 & a_3 & a_2 & 0 & a_0 & 0 \\ 0 & 0 & 0 & 4a_4 & 3a_3 & 2a_2 & 0 & 0 \\ 0 & 0 & 0 & a_4 & a_3 & a_2 & 0 & a_0 \\ 0 & 0 & 0 & 0 & 4a_4 & 3a_3 & 2a_2 & 0 \end{pmatrix},$$

where a_i is coefficient of the term y^i in the polynomial $f(y)$. Let D_k denote the determinant of the submatrix of $\mathrm{Discr}(f)$, formed by the first $2k$ rows and the first $2k$ columns, for $k = 1, \ldots, 4$. So, by straightforward calculation one can see that

$$D_1 = 4(a-1)^2(c^2+(d-1)^2)^2$$

$$\begin{aligned} D_2 &= (a-1)^2(c^2+(d-1)^2)^2(3b^2c^4 - 16(a-1)^2 \\ &\quad \times (d-1)^2(c^2+(d-1)^2)), \end{aligned}$$

$$\begin{aligned} D_3 &= -4(a-1)^4c^2(d-1)^2(c^2+(d-1)^2)^2 \\ &\quad \times (3b^2c^4 - c^2(d-1)^2(16(a-1)^2 - b^2) \\ &\quad - 16(a-1)^2(d-1)^4) \end{aligned}$$

$$\begin{aligned} D_4 &= (27(a-1)b^4c^8 - 32(a-1)^3c^6(d-1)^2 \\ &\quad \times (8(a-1)^2 + 9b^2) - 256(a-1)^3c^4(d-1)^4 \\ &\quad \times ((a-1)^2 + b^2))(1-a)(a-1)^2 \\ &\quad \times (d-1)^4(c^2+(d-1)^2)^2. \end{aligned}$$

Now, we prove that if $D_2 \geq 0$ then $D_3 < 0$. Indeed, $D_2 \geq 0$ is equivalent to

$$b^2 \geq \frac{16(a-1)^2(d-1)^2(c^2+(d-1)^2)}{3c^4}.$$

This implies

$$b^2 > \frac{16(a-1)^2(d-1)^2(c^2+(d-1)^2)}{c^2(3c^2+(d-1)^2)}$$

which is equivalent to $D_3 < 0$.

Now, assume that $\Delta > 0$. The sign list of the sequence $\{D_1, D_2, D_3, D_4\}$ is given by

$$[1, \mathrm{sign}(D_2), \mathrm{sign}(D_3), 1]. \tag{11}$$

From the previous facts it follows that the number of sign changes of the revised sign list of (11) is two. Now, the statement (b) follows as per Theorem 1 [Yang *et al.*, 1996].

Assume that $\Delta < 0$. If $D_2 < 0$ and $D_3 > 0$ then we obtain that $f(y)$ has three pairs of conjugate imaginary roots, which is a contradiction. Hence, if $D_2 < 0$ then $D_3 < 0$. The sign list of the sequence $\{D_1, D_2, D_3, D_4\}$ can have one of the following forms

$$[1,-1,-1,-1], \quad [1,1,-1,-1], \quad [1,1,1,-1] \tag{12}$$

which implies that the number of sign changes of the revised sign list of (12) is one. Now, the statement (c) follows as per Theorem 1 [Yang *et al.*, 1996]. Similarly, one can prove statement (d). ■

Lemma 3. *Assume that* $a < 1$ *and* $d < 1$. *Then for* $\epsilon_1 > 0$ *and* $\epsilon_2 > 0$ *the rectangle*

$$\left[0, \frac{b}{1-a} + \epsilon_1\right] \times \left[0, \frac{c}{1-d} + \epsilon_2\right]$$

is invariant and attracts all solutions of system (1).

Proof. Every solution of system (1) satisfies

$$x_{n+1} \leq ax_n + b, \quad y_{n+1} \leq dy_n + c.$$

The majorant system of (1)

$$\begin{cases} u_{n+1} = au_n + b \\ v_{n+1} = dv_n + c \end{cases}$$

has a solution

$$u_n = C_1 a^{n-1} + \frac{b(1-a^n)}{1-a},$$

$$v_n = C_2 d^{n-1} + \frac{c(1-d^n)}{1-d},$$

which implies that $u_n \to \frac{b}{1-a}$ and $v_n \to \frac{c}{1-d}$ as $n \to \infty$. By using the difference inequality theorem, see [Lakshmikantham & Trigiante, 2002], $x_0 = u_0$ and $y_0 = v_0$ gives $x_n \leq u_n$ and $y_n \leq v_n$ which implies $\limsup_{n\to\infty} x_n \leq \frac{b}{1-a}$ and $\limsup_{n\to\infty} y_n \leq \frac{c}{1-d}$. Thus we conclude that the rectangle

$$\left[0, \frac{b}{1-a} + \epsilon_1\right] \times \left[0, \frac{c}{1-d} + \epsilon_2\right]$$

for $\epsilon_1 > 0$ and $\epsilon_2 > 0$ attracts all solutions. Set

$$U_1 = \frac{b}{1-a} + \epsilon_1, \quad U_2 = \frac{c}{1-d} + \epsilon_2.$$

We have that

$$\begin{aligned} T_1(x,y) &= ax + \frac{by^2}{1+y^2} \\ &\leq ax + b \leq aU_1 + b \leq U_1, \\ T_2(x,y) &= \frac{cx^2}{1+x^2} + dy \\ &\leq dy + c \leq dU_2 + b \leq U_2, \end{aligned}$$

which shows that the set $[0, U_1] \times [0, U_2]$ is invariant. ■

Second, we discuss local stability of the equilibrium solutions.

The equilibrium solutions of the system (1) satisfy the system of Eqs. (5). It follows immediately that the origin is a solution of (5). Geometrically, solutions of (5) are intersections of two orthogonal rational curves (6).

The map associated with the system (1) has the form:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + \dfrac{by^2}{1+y^2} \\ \dfrac{cx^2}{1+x^2} + dx \end{pmatrix}.$$

The Jacobian matrix of T is

$$J_T(x,y) = \begin{pmatrix} a & \dfrac{2by}{(1+y^2)^2} \\ \dfrac{2cx}{(1+x^2)^2} & d \end{pmatrix}. \tag{13}$$

The Jacobian matrix of T evaluated in an equilibrium point with positive coordinates $(\overline{x}, \overline{y})$ has the form:

$$J_T(\overline{x},\overline{y}) = \begin{pmatrix} a & \dfrac{2\overline{x}(1-a)}{\overline{y}(1+\overline{y}^2)} \\ \dfrac{2\overline{y}(1-d)}{\overline{x}(1+\overline{x}^2)} & d \end{pmatrix}. \tag{14}$$

The determinant and trace of (14) are:

$$\begin{aligned} \det(J_T(\overline{x},\overline{y})) &= ad - \frac{2(1-d)}{(1+\overline{x}^2)}\frac{2(1-a)}{(1+\overline{y}^2)}, \\ \operatorname{tr}(J_T(\overline{x},\overline{y})) &= a + d. \end{aligned} \tag{15}$$

The eigenvalues of (14) are

$$\lambda = \frac{(d+a) + \sqrt{(a-d)^2 + 4\dfrac{2(1-d)}{(1+\overline{x}^2)}\dfrac{2(1-a)}{(1+\overline{y}^2)}}}{2},$$

$$\mu = \frac{(d+a) - \sqrt{(a-d)^2 + 4\dfrac{2(1-d)}{(1+\overline{x}^2)}\dfrac{2(1-a)}{(1+\overline{y}^2)}}}{2} \tag{16}$$

with corresponding eigenvectors

$$E_\lambda = \left(\frac{\overline{x}(1+\overline{x}^2)}{2(1-d)\overline{y}}\left(\frac{(a-d) + \sqrt{(a-d)^2 + 4\dfrac{2(1-d)}{(1+\overline{x}^2)}\dfrac{2(1-a)}{(1+\overline{y}^2)}}}{2}\right), 1\right),$$

$$E_\mu = \left(\frac{\overline{x}(1+\overline{x}^2)}{2(1-d)\overline{y}} \left(\frac{(a-d) - \sqrt{(a-d)^2 + 4\frac{2(1-d)}{(1+\overline{x}^2)}\frac{2(1-a)}{(1+\overline{y}^2)}}}{2} \right), 1 \right). \tag{17}$$

It is clear from (16) that λ and μ are real numbers such that $\lambda > \mu$ and $\lambda > 0$.

Lemma 4. *The following conditions hold for the coordinates of the positive equilibrium points of system (1).*

(i) *For $E_{NE}(\overline{x}, \overline{y})$*

$$(\overline{x}^2+1)(\overline{y}^2+1) > 4; \tag{18}$$

(ii) *For $E_{SW}(\overline{x}, \overline{y})$*

$$(\overline{x}^2+1)(\overline{y}^2+1) < 4; \tag{19}$$

(iii) *For $E(\overline{x}, \overline{y})$*

$$(\overline{x}^2+1)(\overline{y}^2+1) = 4. \tag{20}$$

Proof

(i) Let $m_{\mathcal{E}_1}$ be the slope of the tangent to rational equation $\mathcal{E}_1$ at $E_{NE}(\overline{x}, \overline{y})$ and let $m_{\mathcal{E}_2}$ be the slope of the tangent to rational equation $\mathcal{E}_2$ at $E_{NE}(\overline{x}, \overline{y})$. It is clear from geometry that

$$m_{\mathcal{E}_1} > m_{\mathcal{E}_2}.$$

See Fig. 2. It follows that

$$\left.\frac{dy}{dx}\right|_{\mathcal{E}_1}(\overline{x}, \overline{y}) > \left.\frac{dy}{dx}\right|_{\mathcal{E}_2}(\overline{x}, \overline{y})$$

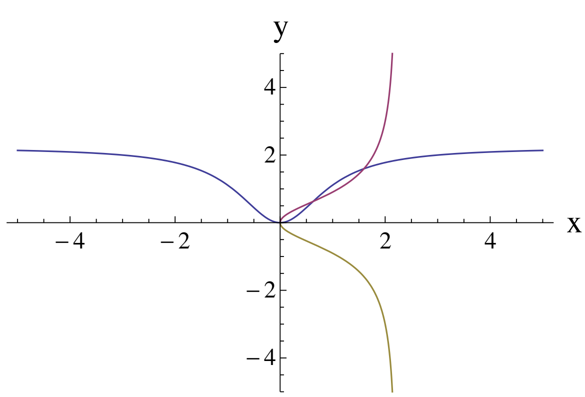


Fig. 2. The curves of the equilibrium points for $b = 2$, $c = 1, a = d = 1/2$.

and in turn

$$\frac{(\overline{y}^2+1)^2(1-a)}{2b\overline{y}} > \frac{2c\overline{x}}{(\overline{x}^2+1)^2(1-d)}$$

which is equivalent to

$$(1-a)(1-d) > \frac{2c\overline{x}2b\overline{y}}{(\overline{x}^2+1)^2(\overline{y}^2+1)^2}.$$

Using the equilibrium condition (5) we may rewrite this as

$$(1-a)(1-d) > \frac{2(1-a)2(1-d)}{(\overline{x}^2+1)(\overline{y}^2+1)}.$$

The proofs for cases (ii) and (iii) are similar and will be omitted. ■

Lemma 5. *$E_0(0,0)$ is locally asymptotically stable.*

Proof. The eigenvalues of (13) evaluated at $E_0(0,0)$ are $\lambda = a$ and $\mu = b$, where $0 < a < 1$ and $0 < b < 1$. ■

Theorem 2. *When system (1) has one positive equilibrium point, $E(\overline{x}, \overline{y})$ is nonhyperbolic of the stable type.*

Proof. We need to show that $\lambda = 1$ and $-1 < \mu < 1$. We will first show that $\lambda = 1$. From (20) and the fact that $a < 1$ and $d < 1$, we have

$$\sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+\overline{x}^2}\frac{2(1-a)}{1+\overline{y}^2}}$$
$$= \sqrt{(a-d)^2 + 4(1-a)(1-d)}$$
$$= \sqrt{(a+d-2)^2} = |a+d-2| = 2-d-a.$$

Therefore

$$\lambda = \frac{(d+a) + \sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+\overline{x}^2}\frac{2(1-a)}{1+\overline{y}^2}}}{2}$$
$$= 1.$$

We will next show $-1 < \mu < 1$. Since by (16) it is clear that $\mu < \lambda$, and we have shown that $\lambda = 1$, it follows that $\mu < 1$. From $a < 1$, $b < 1$ and

$$\sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+\overline{x}^2}\frac{2(1-a)}{1+\overline{y}^2}} = 2 - d - a$$

it follows that

$$\sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+\overline{x}^2}\frac{2(1-a)}{1+\overline{y}^2}} < 2 + d + a.$$

Therefore

$$-1 < \frac{(d+a) - \sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+\overline{x}^2}\frac{2(1-a)}{1+\overline{y}^2}}}{2} = \mu. \qquad \blacksquare$$

Theorem 3. *When system (1) has two positive equilibrium points, $E_{NE}(\overline{x}, \overline{y})$ is locally asymptotically stable.*

Proof. We will first show that $0 < \lambda < 1$. Indeed

$$\lambda = \frac{(d+a) + \sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+\overline{x}^2}\frac{2(1-a)}{1+\overline{y}^2}}}{2} > 0.$$

From (18) and the fact that $a < 1$ and $d < 1$, we have

$$\begin{aligned}\sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+\overline{x}^2}\frac{2(1-a)}{1+\overline{y}^2}} &< \sqrt{(a-d)^2 + 4(1-a)(1-d)} \\ &= \sqrt{(a+d-2)^2} = |a+d-2| \\ &= 2 - d - a.\end{aligned}$$

Therefore

$$\sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+\overline{x}^2}\frac{2(1-a)}{1+\overline{y}^2}} < 2 - d - a$$

and

$$\lambda = \frac{(d+a) + \sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+\overline{x}^2}\frac{2(1-a)}{1+\overline{y}^2}}}{2} < 1.$$

We will next show that $-1 < \mu < 1$.

From (18) and the fact that $a < 1$ and $d < 1$, we have

$$\begin{aligned}\sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+\overline{x}^2}\frac{2(1-a)}{1+\overline{y}^2}} &< \sqrt{(a-d)^2 + 4(1-a)(1-d)} \\ &= \sqrt{(a+d-2)^2} = |a+d-2| \\ &= 2 - d - a < 2 + a + d.\end{aligned}$$

Therefore

$$\sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+\overline{x}^2}\frac{2(1-a)}{1+\overline{y}^2}} < 2 + a + d$$

and

$$-2 < (a+d) - \sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+\overline{x}^2}\frac{2(1-a)}{1+\overline{y}^2}}$$

which yields

$$-1 < \frac{(a+d) - \sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+\overline{x}^2}\frac{2(1-a)}{1+\overline{y}^2}}}{2}$$

and so

$$-1 < \mu.$$

Since $a < 1$ and $d < 1$, we have

$$d + a - 2 < 0 < \sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+\overline{x}^2}\frac{2(1-a)}{1+\overline{y}^2}}.$$

Therefore

$$(d+a) - \sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+\overline{x}^2}\frac{2(1-a)}{1+\overline{y}^2}} < 2$$

and

$$\frac{(d+a) - \sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+\overline{x}^2}\frac{2(1-a)}{1+\overline{y}^2}}}{2} < 1,$$

that is $\mu < 1$. $\blacksquare$

Theorem 4. *When system (1) has two positive equilibrium points, $E_{SW}(\overline{x}, \overline{y})$ is either a repeller, nonhyperbolic of the unstable type, or a saddle point.*

Proof. We need to show that $\lambda > 1$ and either $\mu < -1$, $\mu = -1$, or $-1 < \mu < 1$. We will first show that $\lambda > 1$. From (19) and the fact that $a < 1$ and $d < 1$, we have

$$\begin{aligned}&\sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+\bar{x}^2}\frac{2(1-a)}{1+\bar{y}^2}}\\ &\quad > \sqrt{(a-d)^2 + 4(1-a)(1-d)}\\ &\quad = \sqrt{(a+d-2)^2} = |a+d-2|\\ &\quad = 2-d-a.\end{aligned}$$

Therefore

$$\sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+\bar{x}^2}\frac{2(1-a)}{1+\bar{y}^2}} > 2-d-a$$

and

$$\lambda = \frac{(d+a) + \sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+\bar{x}^2}\frac{2(1-a)}{1+\bar{y}^2}}}{2} > 1.$$

We will next show that $\mu < 1$. Suppose that $\mu \geq 1$. Then

$$\mu = \frac{(d+a) - \sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+\bar{x}^2}\frac{2(1-a)}{1+\bar{y}^2}}}{2} \geq 1$$

which is equivalent to

$$d+a-2 \geq \sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+\bar{x}^2}\frac{2(1-a)}{1+\bar{y}^2}}.$$

This is a contradiction since $a < 1$ and $d < 1$ imply that

$$d+a-2 < 0 < \sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+\bar{x}^2}\frac{2(1-a)}{1+\bar{y}^2}}.$$

Therefore $\mu < 1$.

We will next show that either $\mu < -1$, $\mu = -1$, or $1 > \mu > -1$. Since this covers the remaining parametric space, it is sufficient to show that there exist real values of a, b, c, and d for which each aforementioned value of μ exists.

(i) ***Case*** $\boldsymbol{\mu < -1}$

When

$$a = \frac{8}{100},\quad b = \frac{841}{395},\quad c = \frac{841}{395},\quad d = \frac{8}{100},$$

$$\mu = -\frac{69\sqrt{497341}}{105125} - \frac{21}{25} = -1.30289.$$

(ii) ***Case*** $\boldsymbol{\mu = -1}$

When

$$a = \frac{21}{79},\quad b = \frac{841}{395},\quad c = \frac{841}{395},$$

$$d = \frac{21}{79},\quad \mu = -1.$$

(iii) ***Case*** $\boldsymbol{-1 < \mu < 1}$

When

$$a = \frac{3}{10},\quad b = \frac{841}{395},\quad c = \frac{841}{395},\quad d = \frac{3}{10},$$

$$\mu = -\frac{84\sqrt{697}}{4205} - \frac{2}{5} = -0.927. \qquad \blacksquare$$

Theorem 5. *When system (1) has two positive equilibrium points, the following conditions hold for* $E_{SW}(\bar{x}, \bar{y})$.

(i) $E_{SW}(\bar{x}, \bar{y})$ *is a repeller when*

$$\bar{x}\bar{y} < \frac{4(1-a)^2(1-d)^2}{bc(1+a)(1+d)}.$$

(ii) $E_{SW}(\bar{x}, \bar{y})$ *is nonhyperbolic of the unstable type when*

$$\bar{x}\bar{y} = \frac{4(1-a)^2(1-d)^2}{bc(1+a)(1+d)}.$$

(iii) $E_{SW}(\bar{x}, \bar{y})$ *is a saddle point when*

$$\bar{x}\bar{y} > \frac{4(1-a)^2(1-d)^2}{bc(1+a)(1+d)}.$$

Proof

(i) By (6), when $\bar{x}\bar{y} < \frac{4(1-a)^2(1-d)^2}{bc(1+a)(1+d)}$,

$$(1+a)(1+d) < \frac{4(1-a)(1-d)}{(1+\bar{x}^2)(1+\bar{y}^2)}.$$

It follows that

$$\mu = \frac{(d+a) - \sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+\bar{x}^2}\frac{2(1-a)}{1+\bar{y}^2}}}{2} < -1.$$

The proofs of (ii) and (iii) are similar and will be omitted. $\blacksquare$

3. Global Behavior

In this section, we give the global dynamics results for different local dynamic scenarios of the equilibrium solutions and period-two solutions. The proofs of our results are generic and extend immediately to the case of the general cooperative system (3).

3.1. *The case* $\Delta < 0$

Assume that $\Delta < 0$. By Lemma 2, system (1) has three equilibrium solutions $E_0(0,0)$, $E_{SW}(\overline{x},\overline{y})$ and $E_{NE}(\overline{x},\overline{y})$. By Lemma 5 and Theorem 3 the equilibrium points $E_0(0,0)$ and $E_{NE}(\overline{x},\overline{y})$ are locally asymptotically stable. One can see that $E_0 \ll_{ne} E_{SW} \ll_{ne} E_{NE}$.

Let $\mathcal{B}(E_0)$ be the basin of attraction of $E_0(0,0)$ and $\mathcal{B}(E_{NE})$ be the basin of attraction of E_{NE}.

The following lemma holds.

Lemma 6. *Let* $E_{SW} = (\overline{x}_{SW}, \overline{y}_{SW})$. *The following hold*:

(i) $\text{int}(Q_1(E_{SW})) \subset \mathcal{B}(E_{NE})$.
(ii) $\text{int}(Q_3(E_{SW})) \subset \mathcal{B}(E_0)$.

Proof. By Corollary 1.1 we obtain $\text{int}(Q_3(E_{SW})) \subset \mathcal{B}(E_0)$ and $\text{int}(Q_1(E_{SW}) \cap Q_3(E_{NE})) \subset \mathcal{B}(E_{NE})$. In view of Lemma 3 we have that for $(x_0, y_0) \in \text{int}(Q_1(E_{NE}))$ there exists n_0 such that $T^n(x_0, y_0) \in [\![E_{NE}, (U_1, U_2)]\!]_{ne}$ for all $n > n_0$. Since T is a cooperative map, $T([\![E_{NE}, (U_1, U_2)]\!]_{ne}) \subseteq [\![E_{NE}, (U_1, U_2)]\!]_{ne}$ and E_{NE} is the only equilibrium, we obtain that $[\![E_{NE}, (U_1, U_2)]\!]_{ne} \subset \mathcal{B}(E_{NE})$. Consequently, we obtain that $\text{int}(Q_1(E_{NE})) \subset \mathcal{B}(E_{NE})$. Assume that $(x_0, y_0) \in \text{int}(Q_1(E_{SW}))$. Then, there exists $(\tilde{x}_0, \tilde{y}_0) \in \text{int}(Q_1(E_{SW}) \cap Q_3(E_{NE}))$ such that $(\tilde{x}_0, \tilde{y}_0) \preceq_{ne} (x_0, y_0)$ and $(\tilde{x}_1, \tilde{y}_1) \in \text{int}(Q_1(E_{NE}))$ such that $(x_0, y_0) \preceq_{ne} (\tilde{x}_1, \tilde{y}_1)$. By monotonicity of T we have $T^n(\tilde{x}_0, \tilde{y}_0) \preceq_{ne} T^n(x_0, y_0) \preceq_{ne} T^n(\tilde{x}_1, \tilde{y}_1)$ which implies $T^n(x_0, y_0) \to E_{NE}$ as $n \to \infty$. This implies that $\text{int}(Q_1(E_{SW})) \subset \mathcal{B}(E_{NE})$. The proof of (ii) follows from Corollary 1.1 applied to the ordered interval $[\![E_0, E_{SW}]\!]$. ■

Let $\mathcal{C}_1^+$ denote the boundary of $\mathcal{B}(E_0)$ considered as a subset of $Q_2(E_{SW})$ and $\mathcal{C}_1^-$ denote the boundary of $\mathcal{B}(E_0)$ considered as a subset of $Q_4(E_{SW})$. Also, let $\mathcal{C}_2^+$ denote the boundary of $\mathcal{B}(E_{NE})$ considered as a subset of $Q_2(E_{SW})$ and $\mathcal{C}_2^-$ denote the boundary of $\mathcal{B}(E_0)$ considered as a subset of $Q_4(E_{SW})$. It is easy to see that $E_{SW} \in \mathcal{C}_1^+$, $E_{SW} \in \mathcal{C}_1^-$, $E_{SW} \in \mathcal{C}_2^+$, $E_{SW} \in \mathcal{C}_2^-$, and $T(\mathcal{R}) \subset \text{int}(\mathcal{R})$.

The proof of the following lemmas for cooperative map is the same as the proof of Claims 1 and 2 in [Hadžiabdić *et al.*, 2014] for competitive map, so we skip it (see Fig. 3).

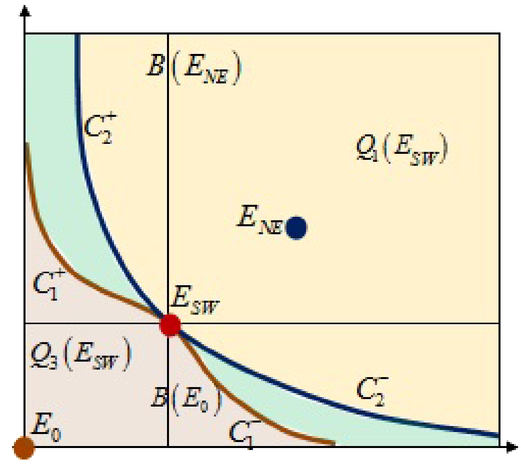


Fig. 3. Visual illustration of Lemmas 7 and 8.

Lemma 7. *Let* $\mathcal{C}_1^+$ *and* $\mathcal{C}_1^-$ *be the sets defined above. Then*

(a) *If* $(x_0, y_0) \in \mathcal{B}(E_0)$ *then* $(x_1, y_1) \in \mathcal{B}(E_0)$ *for all* $(x_1, y_1) \preceq_{ne} (x_0, y_0)$.
(b) *If* $(x_0, y_0) \in \mathcal{C}_1^+ \cup \mathcal{C}_1^-$ *then* $(x_1, y_1) \in \text{int}(\mathcal{B}(E_0))$ *for all* $(x_1, y_1) \ll_{ne} (x_0, y_0)$.
(c) $\mathcal{C}_1^+ \cap \text{int}(Q_2(E_{SW})) \neq \emptyset$ *and* $\mathcal{C}_1^- \cap \text{int}(Q_4 \times (E_{SW})) \neq \emptyset$.
(d) $T(\mathcal{C}_1^+ \cup \mathcal{C}_1^-) \subseteq \mathcal{C}_1^+ \cup \mathcal{C}_1^-$.
(e) $(x_0, y_0), (x_1, y_1) \in \mathcal{C}_1^+ \cup \mathcal{C}_1^- \Rightarrow (x_0, y_0) \ll_{se} (x_1, y_1)$ *or* $(x_1, y_1) \ll_{se} (x_0, y_0)$.
(f) $\mathcal{C}_1^+ \cup \mathcal{C}_1^-$ *is the graph of continuous strictly decreasing function.*

Lemma 8. *Let* $\mathcal{C}_2^+$ *and* $\mathcal{C}_2^-$ *be the sets defined as above. Then*

(a) *If* $(x_0, y_0) \in \mathcal{B}(E_{NE})$ *then* $(x_1, y_1) \in \mathcal{B}(E_{NE})$ *for all* $(x_0, y_0) \preceq_{ne} (x_1, y_1)$.
(b) *If* $(x_0, y_0) \in \mathcal{C}_2^+ \cup \mathcal{C}_2^-$ *then* $(x_1, y_1) \in \text{int}(\mathcal{B} \times (E_{NE}))$ *for all* $(x_0, y_0) \ll_{ne} (x_1, y_1)$.
(c) $\mathcal{C}_2^+ \cap \text{int}(Q_2(E_{SW})) \neq \emptyset$ *and* $\mathcal{C}_2^- \cap \text{int}(Q_4 \times (E_{SW})) \neq \emptyset$.
(d) $T(\mathcal{C}_2^+ \cup \mathcal{C}_2^-) \subseteq \mathcal{C}_2^+ \cup \mathcal{C}_2^-$.
(e) $(x_0, y_0), (x_1, y_1) \in \mathcal{C}_2^+ \cup \mathcal{C}_2^- \Rightarrow (x_0, y_0) \ll_{se} (x_1, y_1)$ *or* $(x_1, y_1) \ll_{se} (x_0, y_0)$.
(f) $\mathcal{C}_2^+ \cup \mathcal{C}_2^-$ *is the graph of continuous strictly decreasing function.*

By Lemmas 8 and 7 it remains to determine the behavior of the orbits of initial conditions (x_0, y_0) such that $(\tilde{x}_0, \tilde{y}_0) \preceq_{ne} (x_0, y_0) \preceq (\overline{x}_0, \overline{y}_0)$ for some $(\tilde{x}_0, \tilde{y}_0) \in \mathcal{C}_1^+ \cup \mathcal{C}_1^-$ and $(\overline{x}_0, \overline{y}_0) \in \mathcal{C}_2^+ \cup \mathcal{C}_2^-$.

Now, we present the global dynamics of system (1) in different parametric regions, based on our numerical simulations. The cases we consider depend on the existence or nonexistence of period-two solutions.

Theorem 6. *If $\Delta < 0$ and a minimal period-two solution does not exist, then system (1) has three equilibrium points $E_0 \ll E_{SW} \ll E_{NE}$, where E_0 and E_{NE} are locally asymptotically stable and E_{SW} is unstable. If E_{SW} is a saddle point then there exist two continuous curves $\mathcal{W}^s(E_{SW})$ and $\mathcal{W}^u(E_{SW})$, both passing through the point E_{SW}, such that $\mathcal{W}^s(E_{SW})$ is a graph of a decreasing function and $\mathcal{W}^u(E_{SW})$ is a graph of an increasing function. The first quadrant of initial condition $Q_1 = \{(x_{-1}, x_0) : x_{-1} \geq 0, x_0 \geq 0\}$ is the union of three disjoint basins of attraction, namely $Q_1 = \mathcal{B}(E_0) \cup \mathcal{B}(E_{SW}) \cup \mathcal{B}(E_{NE})$, where $\mathcal{B}(E_{SW}) = \mathcal{W}^s(E_{SW})$ and*

$$\mathcal{B}(E_0) = \{(x, y) \,|\, (x, y) \preceq_{ne} (x_{E_0}, y_{E_0}) \text{ for some } (x_{E_0}, y_{E_0}) \in \mathcal{W}^s(E_{SW})\}$$

$$\mathcal{B}(E_{NE}) = \{(x, y) \,|\, (x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x, y) \text{ for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(E_{SW})\}.$$

Thus, we have $\mathcal{W}^s(E_{SW}) = \mathcal{C}_1^+ \cup \mathcal{C}_1^- = \mathcal{C}_2^+ \cup \mathcal{C}_2^-$.

Proof. Lemma 2 implies that there exist three equilibrium points namely E_0, E_{SW} and E_{NE} such that $E_0 \ll_{ne} E_{SW} \ll_{ne} E_{NE}$. In this case, E_0 and E_{NE} are locally asymptotically stable and E_{SW} is a saddle point. In view of (13) the map T is strongly cooperative on $[0, \infty)^2$. It follows from the Perron–Frobenius Theorem and a change of variables [Smith, 1998] that, at each point, the Jacobian matrix of a strongly cooperative map has two real and distinct eigenvalues, the larger one in absolute value being positive, and that corresponding eigenvectors may be chosen to point in the direction of the second and first quadrants, respectively. Also, one can show that if the map is strongly cooperative then no eigenvector is aligned with a coordinate axis.

Hence, all conditions of Theorems 1 and 4 from [Kulenović & Merino, 2010] for cooperative map T are satisfied, which yields the existence of the global stable manifold $\mathcal{W}^s(E_{SW})$ and the global unstable manifold $\mathcal{W}^u(E_{SW})$, where $\mathcal{W}^s(E_{SW})$ is passing through the point E_{SW}, and it is a graph of a decreasing function. The curve $\mathcal{W}^u(E_{SW})$ is the graph of an increasing function. Let

$$\mathcal{W}^- = \{(x, y) \,|\, (x, y) \preceq_{ne} (x_{E_0}, y_{E_0}) \text{ for some } (x_{E_0}, y_{E_0}) \in \mathcal{W}^s(E_{SW})\},$$

$$\mathcal{W}^+ = \{(x, y) \,|\, (x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x, y) \text{ for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(E_{SW})\}.$$

Take $(x_0, y_0) \in \mathcal{W}^- \cap [0, \infty)^2$ and $(\tilde{x}_0, \tilde{y}_0) \in \mathcal{W}^+ \cap [0, \infty)^2$. By Theorem 4 [Kulenović & Merino, 2010] we have that there exists $n_0 > 0$ such that, $T^n(x_0, y_0) \in \text{int}(Q_3(E_{SW}))$ and $T^n(\tilde{x}_0, \tilde{y}_0) \in \text{int}(Q_1(E_{SW}))$ for $n > n_0$. The rest of the proof follows from Lemma 6. See Fig. 4(a) for numerical values and the visual illustration. ■

If $\Delta < 0$ and E_{SW} is a nonhyperbolic equilibrium point, then by Theorem 4 we have that $\lambda > 1$ and $\mu = -1$. Based on a series of numerical simulations, we propose the following conjecture. See Fig. 4(b) for numerical values and visual illustration.

Conjecture 1. *If $\Delta < 0$ and the minimal period-two solution does not exist, then system (1) has three equilibrium points $E_0 \ll E_{SW} \ll E_{NE}$, where E_0 and E_{NE} are locally asymptotically stable and E_{SW} is unstable. If E_{SW} is a nonhyperbolic equilibrium point then there exists the continuous curve $\mathcal{C}(E_{SW})$ passing through the point E_{SW}, such that $\mathcal{C}(E_{SW})$ is a graph of decreasing function. The first quadrant of initial condition $Q_1 = \{(x_{-1}, x_0) : x_{-1} \geq 0, x_0 \geq 0\}$ is the union of three disjoint basins of attraction, namely $Q_1 = \mathcal{B}(E_0) \cup \mathcal{B}(E_{SW}) \cup \mathcal{B}(E_{NE})$, where $\mathcal{B}(E_{SW}) = \mathcal{W}^s(E_{SW})$ and*

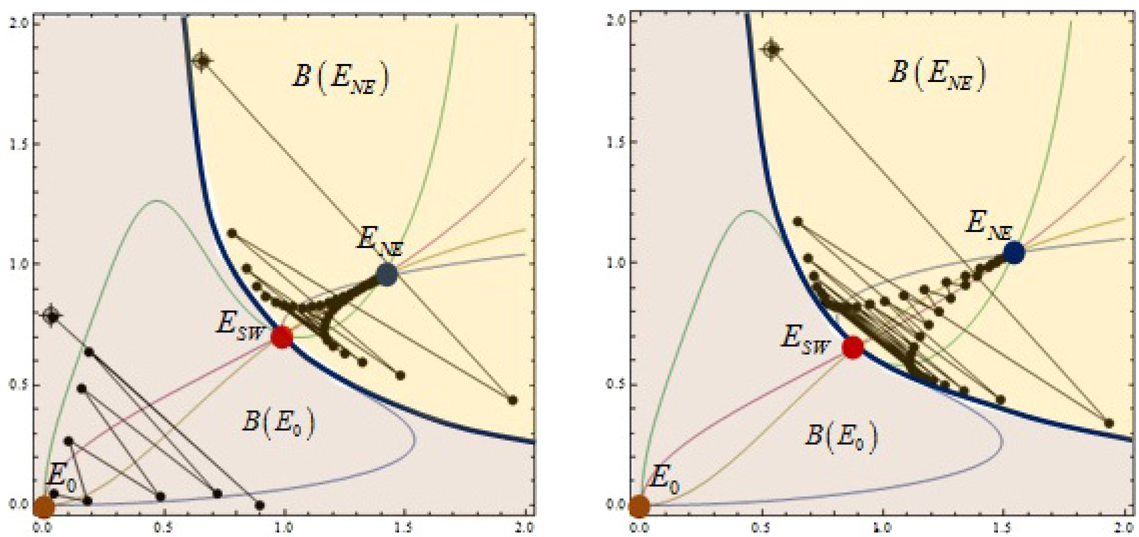
$$\mathcal{B}(E_0) = \{(x, y) \,|\, (x, y) \preceq_{ne} (x_{E_0}, y_{E_0}) \text{ for some } (x_{E_0}, y_{E_0}) \in \mathcal{C}(E_{SW})\}$$

$$\mathcal{B}(E_{NE}) = \{(x, y) \,|\, (x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x, y) \text{ for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{C}(E_{SW})\}.$$

Thus, we have $\mathcal{C}(E_{SW}) = \mathcal{C}_1^+ \cup \mathcal{C}_1^- = \mathcal{C}_2^+ \cup \mathcal{C}_2^-$.

Now, we consider the dynamical scenarios when there exists a minimal period-two solution which is a saddle point. For numerical values and visual illustration, see Fig. 5(a).

Theorem 7. *Assume that $\Delta < 0$ and there exists a minimal period-two solution $\{P_1, T(P_1)\}$ which is a saddle point. Then system (1) has three equilibrium solutions $E_0 \ll E_{SW} \ll E_{NE}$, where E_0*

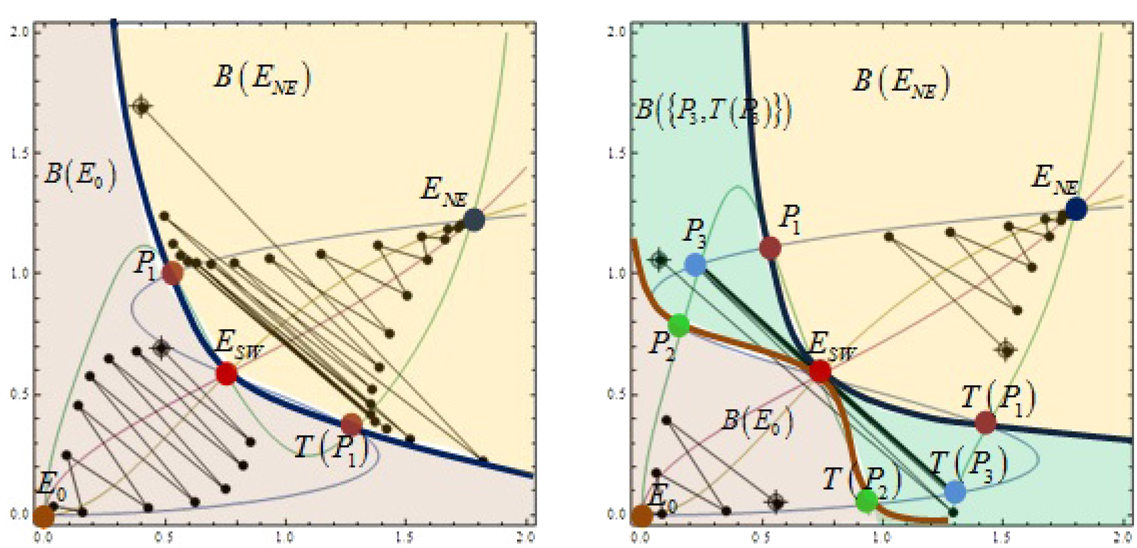


(a) (b)

Fig. 4. (a) Visual illustration of Theorem 6 when $a = 0.21$, $b = 2.34$, $c = 1.43$ and $d = 0.01$, in the case when E_{SW} is a saddle point. (b) Visual illustration of Conjecture 1 when $a = 0.20738$, $b = 2.34$, $c = 1.47$ and $d = 0.01$, in the case when E_{SW} is a nonhyperbolic equilibrium.

and E_{NE} are locally asymptotically stable and E_{SW} is a repeller or nonhyperbolic point. In this case, there exist four continuous curves $\mathcal{W}^s(P_1)$, $\mathcal{W}^s(T(P_1))$, $\mathcal{W}^u(P_1)$, $\mathcal{W}^u(T(P_1))$, where $\mathcal{W}^s(P_1)$, $\mathcal{W}^s(T(P_1))$ are passing through the point E_{SW}, and are graphs of decreasing functions. The curves $\mathcal{W}^u(P_1)$, $\mathcal{W}^u(T(P_1))$ are the graphs of increasing functions and are starting at E_0. Every solution which starts below $\mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1))$ in the North-East ordering converges to E_0 and every solution which starts above $\mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1))$ in the North-East ordering converges to E_{NE}, i.e. $\mathcal{W}^s(P_1) = \mathcal{C}_1^+ = \mathcal{C}_2^+$ and $\mathcal{W}^s(T(P_1)) = \mathcal{C}_1^- = \mathcal{C}_2^-$.

Proof. Since a square T^2 of a cooperative map T is cooperative, all conditions of Theorems 1



(a) (b)

Fig. 5. (a) Visual illustration of Theorem 7 when $a = 0.21$, $b = 2.34$, $c = 1.61$ and $d = 0.02$. The case when E_{SW} is a repeller and $\{P_1, T(P_1)\}$ is a period-two solution which is a saddle point. (b) Visual illustration of Theorem 8 when $a = 0.17$, $b = 2.43$, $c = 1.65$ and $d = 0.01$. The case when E_{SW} is a repeller, $\{P_1, T(P_1)\}$ and $\{P_2, T(P_2)\}$ are period-two solutions which are saddle points and $\{P_3, T(P_3)\}$ is the period-two solution which is locally asymptotically stable.

and 4 from [Kulenović & Merino, 2010] are satisfied, which yields the existence of the global stable manifolds $\mathcal{W}^s(P_1), \mathcal{W}^s(T(P_1))$ and the global unstable manifolds $\mathcal{W}^u(P_1), \mathcal{W}^u(T(P_1))$ where $\mathcal{W}^s(P_1)$, $\mathcal{W}^s(T(P_1))$ are passing through the point E_{SW}, and they are graphs of decreasing functions. Let

$$\mathcal{W}^- = \{(x,y)\,|\,(x,y) \preceq_{ne} (x_{E_0}, y_{E_0}) \text{ for some } (x_{E_0}, y_{E_0}) \in \mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1))\},$$

$$\mathcal{W}^+ = \{(x,y)\,|\,(x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x,y) \text{ for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1))\}.$$

Take $(x_0, y_0) \in \mathcal{W}^- \cap [0,\infty)^2$ and $(\tilde{x}_0, \tilde{y}_0) \in \mathcal{W}^+ \cap [0,\infty)^2$. By Theorem 4 [Kulenović & Merino, 2010] we have that there exist $n_0, n_1 > 0$ such that, $T^n(x_0, y_0) \in \text{int}(Q_3(P_1) \cap Q_3(T(P_1)))$ for $n > n_0$ and $T^n(\tilde{x}_0, \tilde{y}_0) \in \text{int}(Q_1(P_1) \cap Q_1(T(P_1)))$ for $n > n_1$. So, it is enough to prove that $\text{int}(Q_3(P_1) \cap Q_3(T(P_1))) \subseteq \mathcal{B}(E_0)$ and $\text{int}(Q_1(P_1) \cap Q_1(T(P_1))) \subseteq \mathcal{B}(E_{NE})$. Indeed, by Theorem 6 [Kulenović & Merino, 2010] for any $(x_0, y_0) \in \text{int}(Q_3(P_1) \cap Q_3(T(P_1)))$ there exists subsolution $(\tilde{x}_0, \tilde{y}_0)$ (i.e. $T(\tilde{x}_0, \tilde{y}_0) \preceq_{ne} (\tilde{x}_0, \tilde{y}_0)$) such that $(x_0, y_0) \preceq_{ne} (\tilde{x}_0, \tilde{y}_0)$. Since $E_0 \preceq_{ne} T^{2n+2}(\tilde{x}_0, \tilde{y}_0) \preceq_{ne} T^{2n}(\tilde{x}_0, \tilde{y}_0)$ and there is only one period-two solution in $\text{int}(Q_3(P_1) \cap Q_3(T(P_1)))$ we obtain $T^{2n}(\tilde{x}_0, \tilde{y}_0) \to E_0$ as $n \to \infty$. From $T^{2n}(x_0, y_0) \preceq_{ne} T^{2n}(\tilde{x}_0, \tilde{y}_0)$ we have that $T^{2n}(x_0, y_0) \to E_0$ as $n \to \infty$. Since T is a continuous function in the first quadrant we have $T^n(x_0, y_0) \to E_0$ as $n \to \infty$. Similarly, we have that $T^n(x_0, y_0) \to E_{NE}$ as $n \to \infty$ if $(x_0, y_0) \in \text{int}(Q_1(P_1) \cap Q_1(T(P_1)))$. This completes the proof. ■

Now, we consider the dynamical scenario when there exist three minimal period-two solutions. For numerical values and a visual illustration, see Fig. 5(b).

Theorem 8. *Assume that $\Delta < 0$ and there exist three minimal period-two solutions $\{P_i, T(P_i)\}$, $i = 1, 2, 3$, where $\{P_1, T(P_1)\}$ and $\{P_2, T(P_2)\}$ are the saddle points and $\{P_3, T(P_3)\}$ is locally asymptotically stable. Then system (1) has three equilibrium solutions $E_0 \ll E_{SW} \ll E_{NE}$, where E_0 and E_{NE} are locally asymptotically stable and E_{SW} is repeller or nonhyperbolic equilibrium point. In this case there exist four continuous curves $\mathcal{W}^s(P_1)$, $\mathcal{W}^s(T(P_1))$, $\mathcal{W}^s(P_2)$, $\mathcal{W}^s(T(P_2))$ where $\mathcal{W}^s(P_1)$, $\mathcal{W}^s(T(P_1))$, $\mathcal{W}^s(P_2)$, $\mathcal{W}^s(T(P_2))$ are passing through the point E_{SW}, and are graphs of decreasing functions. Every solution which starts below $\mathcal{W}^s(P_2) \cup \mathcal{W}^s(T(P_2))$ in the North-East ordering converges to E_0 and every solution which starts above $W^s(P_1) \cup W^s(T(P_1))$ in the North-East ordering converges to E_{NE}. Every solution which starts above $\mathcal{W}^s(P_2) \cup \mathcal{W}^s(T(P_2))$ and below $\mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1))$ in the North-East ordering converges to $\{P_3, T(P_3)\}$. In other words, the first quadrant of initial condition $Q_1 = \{(x_{-1}, x_0) : x_{-1} \geq 0, x_0 \geq 0\}$ is the union of five disjoint basins of attraction, namely*

$$Q_1 = \mathcal{B}(E_0) \cup \mathcal{B}(\{P_1, T(P_1)\}) \cup \mathcal{B}(\{P_2, T(P_2)\}) \cup \mathcal{B}(\{P_3, T(P_3)\}) \cup \mathcal{B}(E_{NE}),$$

where

$$\mathcal{B}(\{P_1, T(P_1)\}) = \mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1)),$$

$$\mathcal{B}(\{P_2, T(P_2)\}) = \mathcal{W}^s(P_2) \cup \mathcal{W}^s(T(P_2)),$$

$$\mathcal{B}(E_0) = \{(x,y)\,|\,(x,y) \preceq_{ne} (x_{E_0}, y_{E_0}) \textit{ for some } (x_{E_0}, y_{E_0}) \in \mathcal{W}^s(P_2) \cup \mathcal{W}^s(T(P_2))\},$$

$$\mathcal{B}(E_{NE}) = \{(x,y)\,|\,(x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x,y) \textit{ for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(P_1) \cup W^s(T(P_1))\},$$

$$\mathcal{B}(\{P_3, T(P_3)\}) = \{(x,y)\,|\,(x_{E_0}, y_{E_0}) \preceq_{ne} (x,y) \preceq (x_{E_{NE}}, y_{E_{NE}}) \textit{ for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(P_1) \cup W^s(T(P_1)) \textit{ and } (x_{E_0}, y_{E_0}) \in \mathcal{W}^s(P_2) \cup \mathcal{W}^s(T(P_2))\}.$$

Thus, we have $\mathcal{W}^s(P_2) = \mathcal{C}_1^+$, $\mathcal{W}^s(T(P_2)) = \mathcal{C}_1^-$, $\mathcal{W}^s(P_1) = \mathcal{C}_2^+$, and $\mathcal{W}^s(T(P_1)) = \mathcal{C}_2^-$.

Proof. All conditions of Theorems 1 and 4 in [Kulenović & Merino, 2010] for cooperative map T^2 are satisfied, which yields the existence of the global stable manifolds $\mathcal{W}^s(P_1)$, $\mathcal{W}^s(T(P_1))$, $\mathcal{W}^s(P_2)$, $\mathcal{W}^s(T(P_2))$ which are passing through the point E_{SW}, and they are graphs of decreasing functions. Since T is a cooperative map it is easy to see that $P_1 \ll_{ne} P_3 \ll_{ne} P_2$ or $P_2 \ll_{ne} P_3 \ll_{ne} P_1$. Assume $P_2 \ll_{ne} P_3 \ll_{ne} P_1$. As in the proof of Theorem 7 one can see that

$$\mathcal{B}(E_0) = \{(x,y)\,|\,(x,y) \preceq_{ne} (x_{E_0}, y_{E_0}) \text{ for some } (x_{E_0}, y_{E_0}) \in \mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1))\},$$

$$\mathcal{B}(E_{NE}) = \{(x,y)|(x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x,y) \text{ for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(P_2) \cup W^s(T(P_2))\}.$$

So, we assume that $(x_{E_0}, y_{E_0}) \preceq_{ne} (x_0, y_0) \preceq (x_{E_{NE}}, y_{E_{NE}})$ for some $(x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(P_2)$ and $(x_{E_0}, y_{E_0}) \in \mathcal{W}^s(P_1)$. By Theorem 4 in [Kulenović & Merino, 2010] we have that there exists $n_0 > 0$ such that, $T^{2n}(x_0, y_0) \in \text{int}(Q_3(P_1) \cap Q_1(P_2))$ for $n > n_0$. By Corollary 1.1 we get $[[P_2, P_3]] \cup [[P_3, P_1]] \subseteq \mathcal{B}(\{P_3, T(P_3)\})$ which implies that $\text{int}(Q_3(P_1) \cap Q_1(P_2)) = [[P_2, P_1]] \subseteq \mathcal{B}(\{P_3, T(P_3)\})$. If $(x_{E_0}, y_{E_0}) \preceq_{ne} (x_0, y_0) \preceq (x_{E_{NE}}, y_{E_{NE}})$ for some $(x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(T(P_2))$ and $(x_{E_0}, y_{E_0}) \in \mathcal{W}^s(T(P_1))$ then there exists $n_0 > 0$ such that, $T^{2n}(x_0, y_0) \in \text{int}(Q_3(T(P_1)) \cap Q_1(T(P_2)))$ for $n > n_0$. By Corollary 1.1 we get $[[T(P_2), T(P_3)]] \cup [[T(P_3), T(P_1)]] \subseteq \mathcal{B}(\{P_3, T(P_3)\})$ which implies that $\text{int}(Q_3(T(P_1)) \cap Q_1(T(P_2))) = [[T(P_2), T(P_1)]] \subseteq \mathcal{B}(\{P_3, T(P_3)\})$. This completes the proof. ■

Now, we consider two dynamical scenarios when there exists a minimal period-two solution $\{P, T(P)\}$ which is a nonhyperbolic of the stable type (i.e. if μ_1 and μ_2 are eigenvalues of $J_{T^2}(P)$ then $\mu_1 = 1$ and $|\mu_2| < 1$).

Theorem 9. *Assume that $\Delta < 0$ and there exist two minimal period-two solutions $\{P, T(P)\}$ and $\{P_1, T(P_1)\}$, where $\{P, T(P)\}$ is a nonhyperbolic period-two solution of the stable type and $\{P_1, T(P_1)\}$ is a saddle point, and $P \ll_{ne} P_1$ [see Fig. 6(a)]. Then system (1) has three equilibrium points $E_0 \ll E_{SW} \ll E_{NE}$, where E_0 and E_{NE} are locally asymptotically stable and E_{SW} is a repeller or nonhyperbolic equilibrium point. In this case there exist four continuous curves $\mathcal{W}^s(P_1)$, $\mathcal{W}^s \times (T(P_1))$, $\mathcal{C}^s(P)$, $\mathcal{C}^s(T(P))$ where $\mathcal{W}^s(P_1)$, $\mathcal{W}^s \times (T(P_1))$, $\mathcal{C}(P)$, $\mathcal{C}(T(P))$ are passing through the point E_{SW}, which are graphs of decreasing functions. The first quadrant of initial conditions $Q_1 = \{(x_{-1}, x_0) : x_{-1} \geq 0, x_0 \geq 0\}$ is the union of four disjoint basins of attraction, namely*

$$Q_1 = \mathcal{B}(E_0) \cup \mathcal{B}(\{P_1, T(P_1)\}) \cup \mathcal{B}(\{P, T(P)\}) \cup \mathcal{B}(E_{NE}),$$

where

$$\mathcal{B}(\{P_1, T(P_1)\}) = \mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1)),$$

$$\mathcal{B}(E_0) = \{(x, y) \mid (x, y) \preceq_{ne} (x_{E_0}, y_{E_0}) \text{ for some } (x_{E_0}, y_{E_0}) \in \mathcal{C}(P) \cup \mathcal{C}(T(P))\},$$

$$\mathcal{B}(E_{NE}) = \{(x, y) \mid (x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x, y) \text{ for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(P_1) \cup W^s(T(P_1))\},$$

$$\mathcal{B}(\{P, T(P)\}) = \mathcal{C}(P) \cup \mathcal{C}(T(P)) \cup \{(x, y) \mid (x_{E_0}, y_{E_0}) \preceq_{ne} (x, y) \preceq (x_{E_{NE}}, y_{E_{NE}}) \text{ for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(P_1) \cup W^s(T(P_1)) \text{ and } (x_{E_0}, y_{E_0}) \in \mathcal{C}(P) \cup \mathcal{C}(T(P))\}.$$

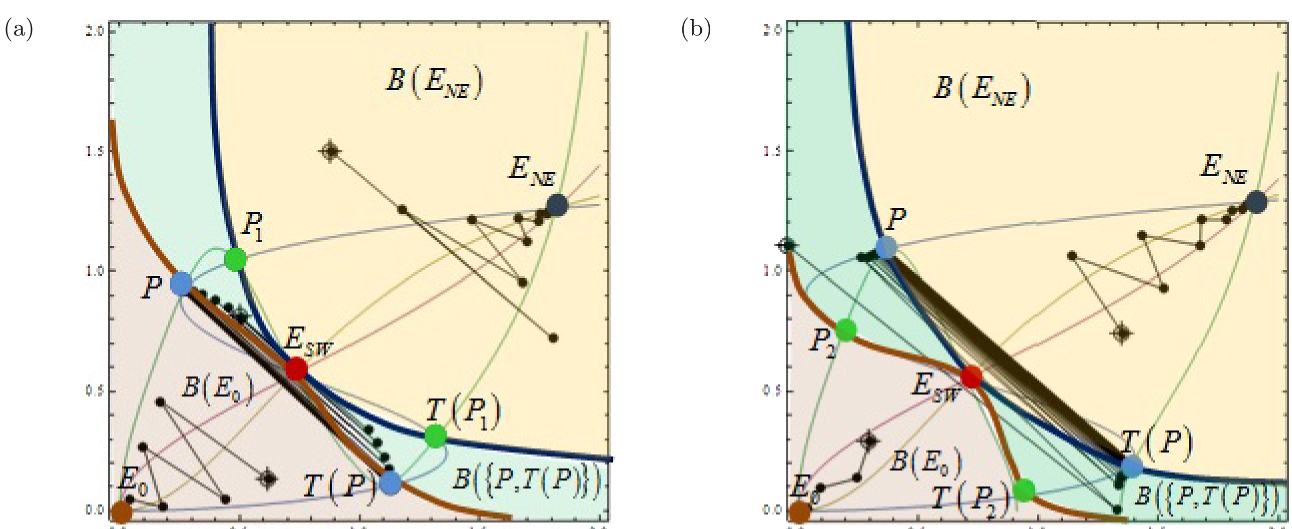


Fig. 6. (a) Visual illustration of Theorem 9 when $a = 0.210703$, $b = 2.34$, $c = 1.638$ and $d = 0.01$. The case when E_{SW} is a repeller, $\{P, T(P)\}$ is the period-two solution which is nonhyperbolic ($\mu_1 = 1$ and $\mu_2 = 0.3797$) and $\{P_1, T(P_1)\}$ is the period-two solution which is a saddle point. (b) Visual illustration of Theorem 10 when $a = 0.2$, $b = 2.43$, $c = 1.61$ and $d = 0.018455$. The case when E_{SW} is a repeller, $\{P, T(P)\}$ is the period-two solution which is nonhyperbolic ($\mu_1 = 1$ and $\mu_2 = 0.567822$) and $\{P_2, T(P_2)\}$ is the period-two solution which is a saddle point.

Thus, we have $\mathcal{C}(P)=\mathcal{C}_1^+$, $\mathcal{C}(T(P))=\mathcal{C}_1^-$, $\mathcal{W}^s(P_1)=\mathcal{C}_2^+$, *and* $\mathcal{W}^s(T(P_1))=\mathcal{C}_2^-$.

Proof. Since $\{P,T(P)\}$ is a nonhyperbolic period-two solution of the stable type and $\{P_1,T(P_1)\}$ is a saddle point, all conditions of Theorems 1 and 4 [Kulenović & Merino, 2010] for cooperative map T^2 are satisfied, which yields the existence of the global stable manifolds $\mathcal{W}^s(P_1),\mathcal{W}^s(T(P_1))$ and invariant curves $\mathcal{C}(P),\mathcal{C}(T(P))$ which are passing through the point E_{SW}, and are graphs of decreasing functions.

Take (x_0,y_0) such that $(x_0,y_0)\preceq_{ne}(x_{E_0},y_{E_0})$ for some $(x_{E_0},y_{E_0})\in\mathcal{C}(P)\cup\mathcal{C}(T(P))$. As per Theorem 4 [Kulenović & Merino, 2010] there exists $n_0>0$ such that, $T^{2n}(x_0,y_0)\in\operatorname{int}(Q_3(P)\cap Q_3(T(P)))$ for $n>n_0$. Since $\operatorname{int}(Q_3(P)\cap Q_3\times(T(P)))\subseteq\operatorname{int}(Q_3(E_{SW}))$ by Lemma 6 we obtain

$$\mathcal{B}(E_0)=\{(x,y)\,|\,(x,y)\preceq_{ne}(x_{E_0},y_{E_0})\text{ for some }(x_{E_0},y_{E_0})\in\mathcal{C}(P)\cup\mathcal{C}(T(P))\}.$$

If (x_0,y_0) such that $(x_{E_{NE}},y_{E_{NE}})\preceq_{ne}(x_0,y_0)$ for some $(x_{E_{NE}},y_{E_{NE}})\in\mathcal{W}^s(P_1)\cup W^s(T(P_1))$, by Theorem 4 [Kulenović & Merino, 2010] there exists $n_0>0$ such that, $T^{2n}(x_0,y_0)\in\operatorname{int}(Q_1(P_1)\cap Q_1(T(P_1)))$ for $n>n_0$. Since $\operatorname{int}(Q_1(P_1)\cap Q_1(T(P_1)))\subseteq\operatorname{int}(Q_1(E_{SW}))$, by Lemma 6 we obtain

$$\mathcal{B}(E_{NE})=\{(x,y)\,|\,(x_{E_{NE}},y_{E_{NE}})\preceq_{ne}(x,y)\text{ for some }(x_{E_{NE}},y_{E_{NE}})\in\mathcal{W}^s(P_1)\cup\mathcal{W}^s(T(P_1))\}.$$

Now, we assume that $(x_{E_0},y_{E_0})\preceq_{ne}(x_0,y_0)\preceq_{ne}(x_{E_{NE}},y_{E_{NE}})$ for some $(x_{E_{NE}},y_{E_{NE}})\in\mathcal{W}^s(P_1)$ and $(x_{E_0},y_{E_0})\in\mathcal{C}(P)$. By Theorem 4 [Kulenović & Merino, 2010] we have that there exists $n_0>0$ such that, $T^{2n}(x_0,y_0)\in\operatorname{int}(Q_3(P_1)\cap Q_1(P))$ for $n>n_0$. By Corollary 1.1 we get $[[P,P_1]]\subseteq\mathcal{B}(\{P,T(P)\})$. Similarly, if $(x_{E_0},y_{E_0})\preceq_{ne}(x_0,y_0)\preceq_{ne}(x_{E_{NE}},y_{E_{NE}})$ for some $(x_{E_{NE}},y_{E_{NE}})\in\mathcal{W}^s(T(P_1))$ and $(x_{E_0},y_{E_0})\in\mathcal{C}(T(P))$, we have that there exists $n_0>0$ such that, $T^{2n}(x_0,y_0)\in\operatorname{int}(Q_3(T(P_1))\cap Q_1(T(P)))$ for $n>n_0$. By Corollary 1.1 we get $[[T(P),T(P_1)]]\subseteq\mathcal{B}(\{P,T(P)\})$. This implies $(x_0,y_0)\in\mathcal{B}(\{P,T(P)\})$. ■

The proof of the following theorem is similar to the proof of Theorem 9 and will be omitted.

Theorem 10. *Assume that* $\Delta<0$ *and there exist two minimal period-two solutions* $\{P,T(P)\}$ *and* $\{P_2,T(P_2)\}$, *where* $\{P,T(P)\}$ *is a nonhyperbolic equilibrium solution of stable type and* $\{P_2,T(P_2)\}$ *is a saddle point, and* $P_2\ll_{ne}P$ *[see Fig. 6(b)]. Then system (1) has three equilibrium solutions* $E_0\ll E_{SW}\ll E_{NE}$, *where* E_0 *and* E_{NE} *are locally asymptotically stable and* E_{SW} *is a repeller or nonhyperbolic equilibrium point. In this case there exist four continuous curves* $\mathcal{W}^s(P_2)$, $\mathcal{W}^s(T(P_2))$, $\mathcal{C}^s(P)$, $\mathcal{C}^s(T(P))$ *where* $\mathcal{W}^s(P_2)$, $\mathcal{W}^s(T(P_2))$, $\mathcal{C}(P)$, $\mathcal{C}(T(P))$ *are passing through the point* E_{SW}, *and are graphs of decreasing functions. The first quadrant of initial condition* $Q_1=\{(x_{-1},x_0):x_{-1}\ge 0,x_0\ge 0\}$ *is the union of four disjoint basins of attraction, namely*

$$Q_1=\mathcal{B}(E_0)\cup\mathcal{B}(\{P_2,T(P_2)\})\cup\mathcal{B}(\{P,T(P)\})\cup\mathcal{B}(E_{NE}),$$

where

$$\mathcal{B}(\{P_2,T(P_2)\})=\mathcal{W}^s(P_2)\cup\mathcal{W}^s(T(P_2)),$$

$$\mathcal{B}(E_0)=\{(x,y)\,|\,(x,y)\preceq_{ne}(x_{E_0},y_{E_0})\textit{ for some }(x_{E_0},y_{E_0})\in\mathcal{W}^s(P_2)\cup\mathcal{W}^s(T(P_2))\},$$

$$\mathcal{B}(E_{NE})=\{(x,y)\,|\,(x_{E_{NE}},y_{E_{NE}})\preceq_{ne}(x,y)\textit{ for some }(x_{E_{NE}},y_{E_{NE}})\in\mathcal{C}(P)\cup\mathcal{C}(T(P))\},$$

$$\mathcal{B}(\{P,T(P)\})=\mathcal{C}(P)\cup\mathcal{C}(T(P))\cup\{(x,y)\,|\,(x_{E_0},y_{E_0})\preceq_{ne}(x,y)\preceq(x_{E_{NE}},y_{E_{NE}})\textit{ for some }(x_{E_{NE}},y_{E_{NE}})\in\mathcal{C}(P)\cup\mathcal{C}(T(P))\textit{ and }(x_{E_0},y_{E_0})\in\mathcal{W}^s(P_2)\cup\mathcal{W}^s(T(P_2))\}.$$

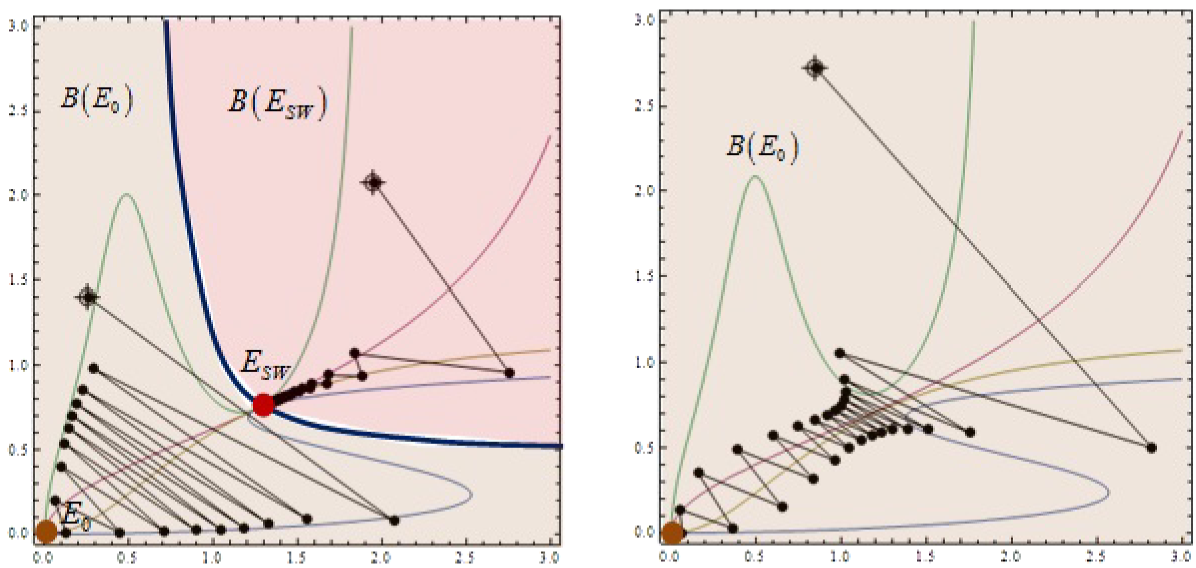
Thus, we have $\mathcal{W}^s(P_2)=\mathcal{C}_1^+$, $\mathcal{W}^s(T(P_2))=\mathcal{C}_1^-$, $\mathcal{C}(P)=\mathcal{C}_2^+$, *and* $\mathcal{C}(T(P))=\mathcal{C}_2^-$.

3.2. *The case* $\Delta\ge 0$

Theorem 11. *Assume that* $\Delta>0$. *Then* $E_0(0,0)$ *is globally asymptotically stable and* $B(E_0)=[0,\infty)^2$.

Proof. By Lemma 6 we have $T([0,\infty)^2)\subseteq[0,U_1]\times[0,U_2]$, so it is sufficient to prove that $T^n(x_0,y_0)\to E_0$ for $(x_0,y_0)\in[0,U_1]\times[0,U_2]$. Take $(x_0,y_0)\in[0,U_1]\times[0,U_2]$. From $T((U_1,U_2))\preceq_{ne}(U_1,U_2)$, since T is a cooperative map, we obtain $T^{n+1}(U_1,U_2)\preceq_{ne}T^n(U_1,U_2)$. By Lemma 2, E_0 is the only equilibrium point, so we have that $T^n(U_1,U_2)\to E_0$ as $n\to\infty$. Further, from $(x_0,y_0)\preceq_{ne}(U_1,U_2)$ we have $E_0\preceq_{ne}T^n(x_0,y_0)\preceq_{ne}T^n(U_1,U_2)$. This implies that $T^n(x_0,y_0)\to E_0$ as $n\to\infty$. The same proof can be accomplished by using Theorem 1.

See Fig. 7(b) for the visual illustration. ■



(a) (b)

Fig. 7. (a) Visual illustration of Theorem 12 — The case when E_{SW} is nonhyperbolic. (b) Visual illustration of Theorem 11 — The case when there exists only one equilibrium point E_0.

Theorem 12. *If $\Delta = 0$ and the minimal period-two solution does not exist, then system (1) has two equilibrium points $E_0 \ll E$, where E_0 is locally asymptotically stable and E is nonhyperbolic and there exists a continuous curve $\mathcal{C}^s(E)$ passing through the point E, such that $\mathcal{C}^s(E)$ is a graph of decreasing function. The first quadrant of initial conditions $Q_1 = \{(x_{-1}, x_0) : x_{-1} \geq 0, x_0 \geq 0\}$ is the union of two disjoint basins of attraction, namely $Q_1 = \mathcal{B}(E_0) \cup \mathcal{B}(E)$, where*

$$\mathcal{B}(E_0) = \{(x, y) \,|\, (x, y) \prec_{ne} (x_{E_0}, y_{E_0}) \text{ for some } (x_{E_0}, y_{E_0}) \in \mathcal{C}^s(E)\},$$

$$\mathcal{B}(E) = \{(x, y) \,|\, (x_E, y_E) \preceq_{ne} (x, y) \text{ for some } (x_E, y_E) \in \mathcal{C}^s(E)\}.$$

Proof. Lemma 2 implies that there exist two equilibrium solutions namely E_0, E such that $E_0 \ll_{ne} E$. By Lemma 5 the equilibrium solution E_0 is locally asymptotically stable. By Theorem 2 the equilibrium solution E is nonhyperbolic of the stable type, i.e. $\lambda = 1$ and $-1 < \mu < 1$. In view of (17) the eigenvector associated with μ is not a coordinate axis. In view of (13) the map T is strongly cooperative on $[0, \infty)^2$. Hence, all conditions of Theorems 1 and 4 [Kulenović & Merino, 2010] for cooperative map T are satisfied, which yields the existence of the invariant curve $\mathcal{C}_E$ which is passing through the point E, and it is the graph of a decreasing function. Let

$$\mathcal{W}^- = \{(x, y) \,|\, (x, y) \preceq_{ne} (x_{E_0}, y_{E_0}) \text{ for some } (x_{E_0}, y_{E_0}) \in \mathcal{C}_E\},$$

$$\mathcal{W}^+ = \{(x, y) \,|\, (x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x, y) \text{ for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{C}_E\}.$$

Take $(x_0, y_0) \in \mathcal{W}^- \cap [0, \infty)^2$ and $(\tilde{x}_0, \tilde{y}_0) \in \mathcal{W}^+ \cap [0, \infty)^2$. By Theorem 4 [Kulenović & Merino, 2010] we have that there exists $n_0 > 0$ such that, $T^n(x_0, y_0) \in \text{int}(Q_3(E))$ and $T^n(\tilde{x}_0, \tilde{y}_0) \in \text{int}(Q_1(E))$ for $n > n_0$. The rest of the proof follows from Lemma 6. See Fig. 7(a) for its visual illustration. ■

Remark 3.1. We believe that Theorems 6–12 can be generalized to a general cooperative system (3) with the same configuration and local character of its equilibrium and period-two solutions.

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