Gazi University Journal of Science
GU J Sci
29(3):615-626 (2016)

On The Application Of Random Walk With Delay And Pareto Distributed Interference Of Chance To An Insurance Model

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Received: 05/01/2016 Revised30/03/2016 Accepted: 06/04/2016

ABSTRACT

In this study, a semi-Markovian random walk with Pareto distributed interference of chance and delay is considered. Some exact formulas for the first four stationary moments of the process are obtained when the random variables which express the discrete interference of chance have Pareto distribution with parameters. The random variables are interpreted as loans which insurance company gets from a bank. With the use of these exact formulas, the third-order asymptotic expansions for the first four stationary moments of the process X(t) are derived when is sufficiently large. Finally, by using Monte-Carlo simulation method, the accuracy of the obtained approximation formulas is tested.

Keywords: *Insurance model; random walk with delay; Pareto distribution; asymptotic expansion; Monte-Carlo simulation method.*

1. INTRODUCTION

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It is known that numerous interesting problems of insurance theory can be expressed and solved by the stochastic processes with a discrete interference of chance. In general, the capital of an insurance company increases by the customer's payment and it decreases by compensating the damage of the accidents that occurring randomly. Insurance companies may maintain their business prosperously for a short time, but this doesn't signify that they will carry on their profit margin in the future. Because, depending on unluckiness, the accidents that happen one after another may lead the insurance company to bankruptcy. If insurance companies don't want to come up against such risks, they have to take some additional precautionary measures.

In this study, we adopt a random walk approach to insurance's problem. So we consider a stochastic model, which can be used for the needs of the insurance theory. This model can be described as follows:

Suppose that the initial capital of an insurance company is equal to . Assume the premiums and claims arrive at the insurance company randomly. , , here , , denote; the random time intervals between two successive customers. The total capital level of the company passes from a state to another by considering , , premiums , claims and jumping in the moments. The random variable expressing premiums and claims can take both positive and negative values. Capital continues its natural variation until random moment which is a first moment of the capital level falls below s. Here the constant is a positive number, interpreted as a control level for the capital of the company. When the case above occurs, working of the insurance company is stopped at the control level s for a random time . Usually, the random variables and are called as delaying time and delaying coefficient for the insurance company to make a decision on additional precautionary measures, respectively. Because of these additional measures, the amount of the company's capital is increased to the level , which is a random variable having a certain distribution in the interval Thus, the first period is completed. Then the insurance company keeps working in a way similar to the previous period.

X(t) process denote the variation of the amount of the capital at time t. This process is a semi-Markovian random walk with a discrete interference of chance.

It is theoretically and practically very important to study the stationary characteristics. It is obvious that these characteristics depend on the moment's of the random variables and . Note that, some important studies on this topic exist in the literature (see, for example, Aliyev R. T. and et al. [1-2]; Anisimov V. V., Artalejo J. R. [3]; Borovkov A.A. [4]; Khaniyev T.A. and et al. [7-8]; Lotov V.I. [9]; Rogozin B.A. [10]; etc.).

Note that, in the studies [8], [2] and [1] the random variable , which describes the discrete interference of chance, have an exponential, triangular and gamma distribution, respectively and the stationary moments of ergodic distribution have been investigated. In this study, unlike [8], [2] and [1], we accept that the random variables , have Pareto distribution with parameters . Pareto distribution was invented by the Italian economist Vilfrido Pareto to describe the distribution of income. Today it has a wide application area from economics to insurance theory and to sociology. The cause of such popularity of Pareto distribution is unlike the other distributions like gamma and exponential distribution Pareto distribution follows power law. A random variable which follows power law can have large values, and they plays an important role for analyzing extreme events. In studies such as insurance and risk theory, such extreme values have a great importance and the ignorance of these extreme events can result with big losses or bankruptcies. Hence Pareto distribution is the best candidate for the interference random variables, which allows us to apply the considered random walk process to insurance theory.

The exact formulas for the first four stationary moments of the process are obtained. In addition to these, third-order asymptotic expansions for the first four stationary moments of the process X(t) are offered, as . Finally, by using Monte-Carlo simulation method, the accuracy of the given approximating formulas is tested.

2. MATHEMATICAL CONSTRUCTION OF THE PROCESS

Let $\{\xi_n\}$, $\{\eta_n\}$, $\{\theta_n\}$, $n \geq 1$ be three independent sequences of random variables defined on any probability space $(\Omega, \Im, P)$, such that variables in each sequence independent and identically distributed. Suppose that ξ_i's and θ_i's take only positive values, η_i's take positive and negative values and their distribution functions are denoted by $\Phi(t)$, $H(u)$ and $F(x)$ respectively. So,

$$\Phi(t) = P\{\xi_1 \leq t\},\ t > 0\ ;$$

$$F(x) = P\{\eta_1 \leq x\},\ x > 0\ ;$$

$$H(u) = P\{\theta_1 \leq u\},\ u > 0\ .$$

Introduce also, sequence of random variables $\{\zeta_n\},\ n \geq 1$, which describes the discrete interference of chance has Pareto distribution with parameters (α, λ)

$$\pi(z) = 1 - \left(\frac{\lambda}{z}\right)^{\alpha},\ z \in [\lambda, \infty)$$

: . Define renewal sequence $\{T_n\}$ and random walk $\{S_n\}$ as follows:

$$T_n = \sum_{i=1}^{n} \xi_i,\qquad S_n = \sum_{i=1}^{n} \eta_i,\qquad T_0 = S_0 = 0,$$

$n = 1, 2, \ldots$ and a sequence of integer valued random variables $\{N_n\}$ as:

$$N_0 = 0,$$

$$N_1 \equiv N(z) = \inf\{n \geq 1:\ z\text{-}S_n < 0\},\ z \geq s\ .$$

$$N_{n+1} = \inf\left\{k \ge 1: \zeta_n - \left(\sum_{i=N_1+N_2+...+N_n+1}^{N_1+N_2+...+N_n+k} \eta_i\right) < 0\right\}$$

$$= \inf\{k \ge 1: S_{N_1+N_2+...+N_n+k} - S_{N_1+N_2+...+N_n} > \zeta_n\},$$

$n = 1,2,...$

Here $s > 0$ and $\inf\{\varnothing\} = +\infty$ is stipulated.

Let $\tau_0 = \gamma_0 = 0$, $\tau_n = T_{N_1+...+N_n}$, $\gamma_n = \tau_n + \theta_n$, $n \ge 1$

and define $\nu(t)$ as: $\nu(t) = \max\{n \ge 0: \ T_n \le t\}$.

We can now construct the desired stochastic process $X(t)$ as follows:

$$X(t) = s + \zeta_n - \left(\sum_{i=N_1+N_2+...+N_n+1}^{\nu(t)} \eta_i\right)$$

$$= s + \zeta_n - (S_{\nu(t)} - S_{N_0+N_1+...N_n}),$$ if

$\gamma_n \le t < \gamma_{n+1}$, $n = 0,1,2,...$; $\zeta_0 = z \in [\lambda, \infty)$.

In this study, the process $X(t)$ will be called "a semi-Markovian random walk with delay and Pareto distributed interference of chance".

The main purpose of this study is to investigate the asymptotic behavior of the stationary moments of the process $X(t)$, as $E(\zeta_1) \to \infty$. For this purpose, we first discuss the ergodicity of the process $X(t)$.

3. PRELIMINARY DISCUSSIONS

Firstly, we can state the following lemma from [1].

Lemma 3.1. Let the initial sequences of the random variables $\{\xi_n\}$, $\{\eta_n\}$, $\{\theta_n\}$ and $\{\zeta_n\},\ n \ge 1$ satisfy the following supplementary conditions:

1) $E\xi_1 < \infty$; 2) $E\theta_1 < \infty$; 3) $0 < E\eta_1 < \infty$; 4) η_1 is non-arithmetic random variable; 5) $E(\eta_1^2) < \infty$; 6) the sequence of the random variables $\{\zeta_n\},\ n \ge 1$, which describes the discrete interference of chance has Pareto distribution with the parameters (α, λ). Then the process $X(t)$ is ergodic and the following relation for ergodic distribution function $Q_X(x)$ holds:

$$Q_X(x) \equiv \lim_{t\to\infty} P\{X(t) \le x\} = \frac{E(A(x,\zeta_1))}{E(N(\zeta_1))}. \tag{3.1}$$

where

$$E(N(\zeta_1)) = \int_0^\infty E(N(z))d\pi(z);\ E(A(x,\zeta_1)) = \int_0^\infty A(x,z)d\pi(z);$$

$$A(x, z) = \sum_{n=0}^\infty a_n(x,z);\ a_n(x,z) = P\left\{z - S_i > 0, i = \overline{1,n}; z - S_n \le x\right\}$$

, $x > 0;\ z > \lambda$.

Remark 3.1. Let's now put $\varphi_X(u) \equiv \lim_{t\to\infty} E\{\exp(iuX(t))\}$, $u \in R$. Using the basic identity for the random walks (see, Feller W., [5], p.514) and Lemma 3.1, we obtain the following Lemma 3.2.

Lemma 3.2. Let the conditions of Lemma 3.1 be satisfied. Then for $u \in R/\{0\}$, the characteristic function $\varphi_X(u)$ of the ergodic distribution of the process $X(t)$ can be expressed by means of the characteristics of the pair $(N(x), S_{N(x)})$ and the random variable η_1 as follows:

$$\varphi_X(u) = \frac{e^{ius}}{EN(\zeta_1)+K}\int_0^\infty x^{-(\alpha+1)} e^{iuz} \frac{\varphi_{S_{N(x)}}(-u)-1}{\varphi_\eta(-u)-1} dx$$

$$+ \frac{K e^{ius}}{EN(\zeta_1)+K}\int_0^\infty x^{-(\alpha+1)} e^{iuz} \varphi_{S_{N(x)}}(-u)dx, \tag{3.2}$$

where $EN(\zeta_1) = \alpha\lambda^\alpha \int_0^\infty x^{-(\alpha+1)} EN(x)dx$;

$\varphi_{S_{N(x)}}(-u) = E\exp(-iu S_{N(x)})$;

$\varphi_\eta(-u) = E\exp(-iu\,\eta_1)$; $K = E\theta_1 / E\xi_1$.

4. EXACT FORMULAS FOR THE FIRST FOUR MOMENTS OF THE ERGODIC DISTRIBUTION OF PROCESS X(T)

The aim of this section is to express the first four moments of the ergodic distribution of the process $X(t)$ by the characteristics of the boundary functional $S_{N(x)}$ and the random variable η_1. For this aim, introduce the following notations:

$$m_k = E(\eta_1^k),\ M_k(x) = E(S_{N(x)}^k),\quad m_{k1} = \frac{m_k}{m_1},\ M_{k1}(x) = \frac{M_k(x)}{M_1(x)},\quad k = \overline{1,5};\ x \ge 0;$$

$$E(\zeta_1^n M_k(\zeta_1)) = \alpha\lambda^\alpha \int_0^\infty x^{-(\alpha+n-1)} M_k(x)dx,\quad n = \overline{0,4};\ e_k = E(\zeta_1^k),\ k = \overline{1,4};$$

and for the shortness of the expressions we put:

$$E(X^k) \equiv \lim_{t\to\infty} E((X(t))^k),\quad k = \overline{1,4} \text{ and } \overline{X}(t) = X(t) - s.$$

Now, we can state the following proposition from [2].

Proposition 4.1. Let the conditions of Lemma 3.1 be satisfied and also $E|\eta_1|^5 < \infty$. Then the first four moments of the ergodic distribution of the process $\overline{X}(t)$ exist and can be expressed by means of the characteristics of the boundary functional $S_{N(x)}$ and the random variable η_1 as follows:

$$E(\overline{X}) = \frac{1}{E(M_1(\zeta_1)) + Km_1}\Big\{E(\zeta_1 M_1(\zeta_1) - \frac{1}{2}E(M_2(\zeta_1)) + \frac{1}{2}(m_{21} - 2Km_1)E(M_1(\zeta_1)) + Km_1 e_1\Big\}; \tag{4.1}$$

$$E(\overline{X}^2) = \frac{1}{E(M_1(\zeta_1)) + Km_1}\Big\{E(\zeta_1^2 M_1(\zeta_1) - E(\zeta_1 M_2(\zeta_1)) + \frac{1}{3}E(M_3(\zeta_1)) + m_{21}\Big(E(\zeta_1 M_1(\zeta_1)) - \frac{1}{2}E(M_2(\zeta_1))\Big) + Km_1\big(E(M_2(\zeta_1)) - 2E(\zeta_1(M_1(\zeta_1))\big) + A_1 E(M_1(\zeta_1)) + Km_1 e_2\Big\}; \tag{4.2}$$

$$E(\overline{X}^3) = \frac{1}{E(M_1(\zeta_1)) + Km_1}\Big\{E(\zeta_1^3 M_1(\zeta_1) - \frac{3}{2}E(\zeta_1^2 M_2(\zeta_1)) + E(\zeta_1 M_3(\zeta_1)) - \frac{1}{4}E(M_4(\zeta_1)) + \frac{3}{2}(m_{21} - 2Km_1)\big(E(\zeta_1^2 M_1(\zeta_1) - E(\zeta_1 M_2(\zeta_1))\big) + \frac{1}{2}(m_{21} - 2Km_1)E(M_3(\zeta_1)) + 3A_1 E(\zeta_1 M_1(\zeta_1)) - \frac{3}{2}A_1 E(M_2(\zeta_1)) + 3A_2 E(M_1(\zeta_1)) + Km_1 e_3\Big\}; \tag{4.3}$$

$$E(\overline{X}^4) = \frac{1}{E(M_1(\zeta_1)) + Km_1}\Big\{E(\zeta_1^4 M_1(\zeta_1) - 2E(\zeta_1^3 M_2(\zeta_1)) + 2E(\zeta_1^2 M_3(\zeta_1)) - E(\zeta_1 M_4(\zeta_1)) + (m_{21} - 2Km_1)\Big(2E(\zeta_1^3)M_1(\zeta_1)) - 3E(\zeta_1^2 M_2(\zeta_1)) + 2E(\zeta_1 M_3(\zeta_1)) + \frac{1}{2}E(M_4(\zeta_1))\Big) + \frac{1}{5}E(M_5(\zeta_1)) + 6A_1\Big(E(\zeta_1^2 M_1(\zeta_1)) - E(\zeta_1 M_2(\zeta_1)) + \frac{1}{3}E(M_3(\zeta_1))\Big)$$

$$+6A_2\left(2E(\zeta_1 M_1(\zeta_1)) - E(M_2(\zeta_1))\right) + 3A_3 E(M_1(\zeta_1)) + Km_1 e_4\Big\}; \qquad (4.4)$$

where $A_1 = \frac{m_{21}}{2}$; $A_2 = \frac{m_{21}^2}{2} - \frac{m_{31}}{3}$; $A_3 = \frac{m_{41}}{12} - \frac{m_{31}m_{21}}{3} + \frac{m_{21}^3}{4}$;

$$A_4 = \frac{m_{21}^4}{4} - \frac{m_{31}m_{21}^2}{2} + \frac{m_{41}m_{21}}{6} + \frac{m_{31}^2}{9} - \frac{m_{51}}{30}; \; e_k = E(\zeta_1^k), \;\; k = \overline{1,4}; \; K = E\theta_1 / E\xi_1.$$

5. THIRD-ORDER ASYMPTOTIC EXPANSIONS FOR THE FIRST FOUR MOMENTS OF THE ERGODIC DISTRIBUTION

In this section, we will obtain asymptotic expansions for the first four moments of the ergodic distribution of the process $X(t)$. For this aim, we will the use the ladder variables of the random walk $S_n = \sum_{i=1}^{n} \eta_i, \; n \geq 1$, with initial state $S_0 = 0$.

Let $\nu_1^+ = \min\{n \geq 1 : S_n > 0\}$, $\chi_1^+ = S_{\nu_1^+} = \sum_{i=1}^{\nu_1^+} \eta_i$.

Note that, the random variables ν_1^+ and χ_1^+ are called the first strict ascending ladder epoch and ladder height of the random walk $\{S_n\}, \; n \geq 0$, respectively (see, Feller W., [5], p.391).

Now, we state the following auxiliary lemma, by using Lemma 5.1 in [7]:

Lemma 5.1. Suppose that the first three moments of the random variable χ^+ are finite. Then we can write the following third-order asymptotic expansions for the first five moments of boundary functional $S_{N(x)}$, as $\lambda \to 0$:

$$E(M_1(\zeta_1)) = E(\zeta_1) + \frac{1}{2}\mu_{21} + o(\lambda), \qquad (5.1)$$

$$E(M_2(\zeta_1)) = E(\zeta_1^2) + \mu_{21}E(\zeta_1) + \frac{1}{3}\mu_{31} + o(1), \qquad (5.2)$$

$$E(M_3(\zeta_1)) = E(\zeta_1^3) + \frac{3}{2}\mu_{21}E(\zeta_1^2) + \mu_{31}E(\zeta_1) + o\left(\frac{1}{\lambda}\right), \qquad (5.3)$$

$$E(M_4(\zeta_1)) = E(\zeta_1^4) + 2\mu_{21}E(\zeta_1^3) + 2\mu_{31}E(\zeta_1^2) + o\left(\frac{1}{\lambda^2}\right), \qquad (5.4)$$

$$E(M_5(\zeta_1)) = E(\zeta_1^5) + \frac{5}{2}\mu_{21}E(\zeta_1^4) + \frac{10}{3}\mu_{31}E(\zeta_1^3) + o\left(\frac{1}{\lambda^3}\right). \qquad (5.5)$$

where $\mu_k = E\left(\chi^+\right)^k; \mu_k = \mu_k / \mu_1$; $M_k(x) = E(S_{N(x)}^k)$; $k = \overline{1,5}$.

Moreover, let us give the following key lemma.

Lemma 5.2. Let g(x) ($g : R^+ \to R$) be a bounded function, Lebesque measurable function and $\lim_{x \to 0^+} \frac{g(x)}{x^{\alpha+1}} = c, \; \alpha > 0, \; c \in R$. Then for each $\alpha > 0$ the following relation holds: $\lim_{\lambda \to 0^+} \int_0^\infty g\left(\frac{\lambda}{t^{1/\alpha}}\right)dt = 0$.

Proof. If we use transformation of $x = \frac{\lambda}{t^{1/\alpha}}$ on $\lim_{\lambda \to 0^+} \int_0^\infty g(\frac{\lambda}{t^{1/\alpha}})dt$ we get

$$\lim_{\lambda \to 0^+} \lambda^\alpha \alpha \int_0^1 x^{-(\alpha+1)} g(x)dx + \lim_{\lambda \to 0^+} \lambda^\alpha \alpha \int_1^\infty x^{-(\alpha+1)} g(x)dx \tag{5.6}$$

For the proof the sum in (5.6) have to be equal to zero. Firstly, we show that

$$\int_0^1 x^{-(\alpha+1)} g(x)dx < \infty .$$

Under the conditions of Lemma 5.2, for the any $\varepsilon > 0, \exists \delta \equiv \delta(\varepsilon) > 0$ exist such that for any $0 < x < \delta$, the inequality

$$\left| \frac{g(x)}{x^{\alpha+1}} - c \right| < \varepsilon, \quad for \;\; all \;\; x \tag{5.7}$$

holds.

From (5.7), it is obtained that $-\varepsilon < \frac{g(x)}{x^{\alpha+1}} - c < \varepsilon, \;\; 0 < x < \delta$. Namely, for all x $(c-\varepsilon)x^{\alpha+1} < g(x) < (c+\varepsilon)x^{\alpha+1}, \;\; 0 < x < \delta$.

If we choose

$$\varepsilon = 1, \; \Delta = \min\left\{ \frac{1}{2}, \delta \right\} \tag{5.8}$$

We get

$$(c-1)x^{\alpha+1} < g(x) < (c+1)x^{\alpha+1}, \;\; 0 < x < \Delta \tag{5.9}$$

From (5.9), we see that

$$\left| \int_0^1 x^{-(\alpha+1)} g(x)dx \right| \le \int_0^1 \left| \frac{g(x)}{x^{\alpha+1}} \right| dx \tag{5.10}$$

We can write (5.10) as the following:

$$\left| \int_0^1 x^{-(\alpha+1)} g(x)dx \right| \le \int_0^\Delta \left| \frac{g(x)}{x^{\alpha+1}} \right| dx + \int_\Delta^1 \left| \frac{g(x)}{x^{\alpha+1}} \right| dx \tag{5.11}$$

Now, we show that $\int_\Delta^1 \left| \frac{g(x)}{x^{\alpha+1}} \right| dx < \infty$. Since $g(x)$ is bounded, $\exists M$ so that $|g(x)| \le M$. From this, it is obtained that,

$$\int_\Delta^1 \left| \frac{g(x)}{x^{\alpha+1}} \right| dx \le M \int_\Delta^1 x^{-(\alpha+1)} dx = M \left. \frac{x^{-\alpha}}{-\alpha} \right|_\Delta^1 = \frac{M}{-\alpha}(1-\Delta^{-\alpha}) < \infty \tag{5.12}$$

In addition, from (5.8) we get

$$\left| \frac{g(x)}{x^{\alpha+1}} \right| \le \left| \frac{g(x)}{x^{\alpha+1}} - c \right| + |c| = 1 + |c|, \;\; 0 < x < \Delta \tag{5.13}$$

If we use (5.13), it becomes $\int_0^{\Delta}\left|\frac{g(x)}{x^{\alpha+1}}\right|dx \le \int_0^{\Delta}(1+|c|)dx = (1+c)\Delta < \infty$. So

$$\int_0^{\Delta}\left|\frac{g(x)}{x^{\alpha+1}}\right|dx < \infty \tag{5.14}$$

From (5.12) and (5.14), it is concluded that;

$$\lim_{\lambda\to 0^+}\lambda^{\alpha}\int_0^1 x^{-(\alpha+1)}g(x)dx = 0 \tag{5.15}$$

Thus, it is obtained that first part of the sum of limits in (5.6) is equal to zero. Secondly, we show that the second part is also equal to zero. For this, since $g(x)$ is a bounded function, for all $x \in R,\ |g(x)| \le M$ where $\exists M$. Using this, it gives

$$\left|\int_1^{\infty} x^{-(\alpha+1)}g(x)dx\right| \le \int_1^{\infty} x^{-(\alpha+1)}|g(x)|dx \le M\int_1^{\infty} x^{-(\alpha+1)}dx \le \frac{M}{\alpha} < \infty$$, for $\alpha > 0$

So we get that,

$$\left|\int_1^{\infty} x^{-(\alpha+1)}g(x)dx\right| < \infty \tag{5.16}$$

From (5.16)

$$\lim_{\lambda\to 0^+}\lambda^{\alpha}\int_1^{\infty} x^{-(\alpha+1)}g(x)dx = 0 \tag{5.19}$$

Thus, it is obtained that second part of the sum of limits in (5.6) is also equal to zero. From (5.15) and (5.17)

$$\lim_{\lambda\to 0^+}\int_0^{\infty} g\left(\frac{\lambda}{t^{1/\alpha}}\right)dx = 0$$.

The proof of the Lemma 5.2 is completed. ■

Let's give the following corollary, which proof is similar to proof of Lemma 5.2.

Corollary 5.1. Let $g(x)$ be defined as in Lemma 5.2 and the function $R_n(x)$ be defined as $R_n(x) \equiv x^n g(x),\ n = -1, 0, 1, 2, \ldots$ Then each $\alpha > 0$, the following asymptotic relation is true, when $\lambda \to 0$:

$$\int_0^{\infty} R_n\left(\frac{\lambda}{t^{1/\alpha}}\right)dt = o\left(\lambda^n\right)$$.

Now, we can state the first main result of this section as follows:

Theorem 5.1. Let the conditions of Proposition 4.1 be satisfied. Then the following asymptotic expansion can be written for the first four moments of the ergodic distribution of the process $\overline{X}(t)$, for each $\alpha > 6$, as $\lambda \to 0$:

$$E(\overline{X}) = D_{21}(\alpha)\lambda + B_{11} + B_{12}\frac{1}{\lambda} + o\left(\frac{1}{\lambda}\right) \tag{5.20}$$

$$E(\overline{X}^2) = D_{31}(\alpha)\lambda^2 + B_{21}(\alpha)\lambda + B_{22}(\alpha) + o(1) \tag{5.21}$$,

$$E(\bar{X}^3) = D_{41}(\alpha)\lambda^3 + B_{31}(\alpha)\lambda^2 + B_{32}(\alpha)\lambda + o(\lambda), \tag{5.22}$$

$$E(\bar{X}^4) = D_{51}(\alpha)\lambda^4 + B_{41}(\alpha)\lambda^3 + B_{42}(\alpha)\lambda^2 + o(\lambda^2), \tag{5.23}$$

where $D_k(\alpha) = \frac{\alpha}{\alpha - k}$, $\alpha > k$; $D_{k1}(\alpha) = \frac{D_k(\alpha)}{kD_1(\alpha)}$, $k = \overline{1,5}$,

$$B_{11}(\alpha) = \frac{1}{2}\left[m_{21} - \frac{D_{21}(\alpha)}{D_1(\alpha)}(\mu_{21} + 2Km_1) \right],$$

$$B_{12}(\alpha) = \frac{D_{21}(\alpha)}{4D_1^2(\alpha)}\left(\mu_{21} + 2Km_1\right)^2 - \frac{1}{6D_1(\alpha)}(\mu_{31} + 3\mu_{21}m_1K + 3m_2K),$$

$$B_{21}(\alpha) = \frac{1}{2}\left[2D_{21}(\alpha)m_{21} - \frac{D_{31}(\alpha)}{D_1(\alpha)}(\mu_{21} + 2Km_1) \right],$$

$$B_{22}(\alpha) = \frac{D_{31}(\alpha)}{4D_1^2(\alpha)}(\mu_{21} + 2Km_1)^2 - \frac{D_{21}(\alpha)}{2D_1(\alpha)}(m_{21}\mu_{21} - 2Km_2) + \frac{3m_{21}^2 - 2m_{31}}{6},$$

$$B_{31}(\alpha) = \frac{3D_{31}(\alpha)}{2}m_{21} - \frac{D_{41}(\alpha)}{2D_1(\alpha)}(\mu_{21} + 2Km_1),$$

$$B_{32}(\alpha) = \frac{D_{21}(\alpha)}{2}(3m_{21}^2 - 2m_{31}) + \frac{2D_{41}(\alpha)}{3D_1^2(\alpha)}(\mu_{21}^2 + \frac{3}{2}\mu_{21}Km_1 + \frac{3}{2}K^2m_1^2)$$
$$- \frac{3D_{31}(\alpha)}{4D_1(\alpha)}(m_{21}\mu_{21} + 2Km_2),$$

$$B_{41}(\alpha) = 6D_{41}(\alpha)(m_{21} - \mu_{21} - \frac{4}{3}Km_1) - \frac{D_{51}(\alpha)}{2D_1(\alpha)}(\mu_{21} + 2Km_1),$$

$$B_{42}(\alpha) = D_{31}(\alpha)(3m_{21}^2 - 2m_{31} + 6m_{21}\mu_{21} - 3\mu_{31} - 12Km_1\mu_{21}) + \frac{D_{51}(\alpha)}{4D_1^2(\alpha)}(\mu_{21} + 2Km_1)^2$$
$$- \frac{D_{41}(\alpha)}{D_1(\alpha)}(3m_{21}\mu_{21} - 3\mu_{21}^2 - 6Km_2 - 10Km_1\mu_{21} - 8K^2m_1^2).$$

Proof. Firstly, we obtain the asymptotic expansion for the expectation of the ergodic distribution of the process $\bar{X}(t)$, as $\lambda \to 0$. For this aim, the exact formula (4.1) was obtained for $E(\bar{X})$ in Proposition 4.1. For the shortness, we put

$$E(\bar{X}) = K(\lambda)J_3(\lambda), \tag{5.24}$$

where $K(\lambda) = \frac{1}{E(M_1(\zeta_1)) + Km_1}$; $J_3(\lambda) = J_1(\lambda) + J_2(\lambda)$;

$J_1(\lambda) = E\left(\zeta_1 M_1(\zeta_1)\right) - \frac{1}{2}E\left(M_2(\zeta_1)\right)$; $J_2(\lambda) = \frac{1}{2}(m_{21} - 2Km_1)E(M_1(\zeta_1)) + Km_1e_1.$

By using Lemma 5.1 and Corollary 5.1, we get the following expansion, as $\lambda \to 0$:

$$J_1(\lambda) = \frac{D_2(\alpha)}{2}\lambda^2 - \frac{\mu_{31}}{6} + o(1) \quad (5.25)$$

Using Lemma 5.1 and Corollary 5.1, we obtain the following asymptotic expansion for $J_2(\lambda)$, as $\lambda \to 0$:

$$J_2(\lambda) = \frac{D_1(\alpha)}{2} m_{21}\lambda + \frac{1}{4}(m_{21} - 2Km_1)\mu_{21} + o(\frac{1}{\lambda}) \quad (5.26)$$

By using asymptotic expansions (5.25) and (5.26), we get:

$$J_3(\lambda) = D_2(\alpha)\lambda^2\left[\frac{1}{2} - \frac{m_{21}D_1(\alpha)}{2D_2(\alpha)}\frac{1}{\lambda} + \frac{1}{D_2(\alpha)}\left(\frac{1}{4}(m_{21} - 2Km_1)\mu_{21} - \frac{1}{6}\mu_{31}\right)\frac{1}{\lambda^2} + o(\frac{1}{\lambda^2})\right]. \quad (5.27)$$

Analogically, we calculate:

$$K(\lambda) = \frac{1}{D_1(\alpha)}\frac{1}{\lambda}\left[1 - \frac{\mu_{21} + 2Km_1}{2D_1(\alpha)}\frac{1}{\lambda} + \left(\frac{\mu_{21} + 2Km_1}{2D_1(\alpha)}\right)^2\frac{1}{\lambda^2} + o(\frac{1}{\lambda^2})\right]. \quad (5.28)$$

Taking into account the asymptotic expansions (5.27) and (5.28), we obtain the following asymptotic expansion, as $\lambda \to 0$:

$$K(\lambda)J_3(\lambda) = D_{21}\lambda - \frac{1}{2}\left[m_{21} - (\mu_{21} + 2Km_1)\frac{D_2(\alpha)}{D_1^2(\alpha)}\right] + \left[(\mu_{21} + 2Km_1)^2\frac{D_2(\alpha)}{8D_1^3(\alpha)}\right.$$
$$\left. - (\mu_{31} + 3\mu_{21}Km_1 + 3Km_2)\frac{1}{6D_1(\alpha)}\right]\frac{1}{\lambda} + o(\frac{1}{\lambda}). \quad (5.29)$$

Substituting (5.29) in (5.24), we finally get the asymptotic expansion (5.20) for $E(\overline{X})$, as $E(\zeta_n) \to \infty$.

Now, we can analogically derive the asymptotic expansion for the second moment of the ergodic distribution of the process $\overline{X}(t)$. For this aim, the exact formula (4.2) was obtained for $E(\overline{X}^2)$ in Theorem 4.1. For the shortness, we put

$$E(\overline{X}^2) = R(\lambda)J_6(\lambda), \quad (5.30)$$

where $J_6(\lambda) = J_4(\lambda) + J_5(\lambda)$; $J_4(\lambda) = E\left(\zeta_1^2 M_1(\zeta_1)\right) - E\left(\zeta_1 M_2(\zeta_1)\right) + \frac{1}{3}E(M_3(\zeta_1))$;

$$J_5(\lambda) = (m_{21} - 2Km_1)E(\zeta_1 M_1(\zeta_1)) - \frac{1}{2}(m_{21} - 2Km_1)E(M_2(\zeta_1)) + A_1E(M_1(\zeta_1)) + Km_1e_2.$$

Using Lemma 5.1 and Corollary 5.1, we obtain the following asymptotic expansion for $J_4(\lambda)$, as $\lambda \to 0$:

$$J_4(\lambda) = \frac{D_3(\alpha)}{3}\lambda^3 + o(\frac{1}{\lambda}) \quad (5.31)$$

Taking Lemma 5.1 and Corollary 5.1 into account, we write the following asymptotic expansion for $J_5(\lambda)$, as $\lambda \to 0$:

$$J_5(\lambda) = \frac{m_{21}}{2}D_2(\alpha)\lambda^2 + A_1D_1(\alpha)\lambda + \frac{A_1\mu_{21}}{2} - \frac{(m_{21} - Km_1)}{6}\mu_{31} + o(1). \quad (5.32)$$

Using asymptotic expansions (5.31) and (5.32), we obtain the following asymptotic expansion for $J_6(\lambda)$, as $\lambda \to 0$:

$$J_6(\lambda) = \frac{D_3(\alpha)}{3}\lambda^3\left[1 + \frac{3m_{21}}{2}\frac{D_2(\alpha)}{D_3(\alpha)}\frac{1}{\lambda} + A_1\frac{D_1(\alpha)}{D_3(\alpha)}\frac{1}{\lambda^2} + \left(\frac{A_1\mu_{21}}{2} - \frac{(m_{21} - Km_1)}{6}\mu_{31}\right)\frac{1}{D_3(\alpha)} + o(1)\right]. \tag{5.33}$$

Substituting asymptotic expansions (5.28) and (5.33) in the formula (5.30), and carrying out the corresponding calculation, we finally get the asymptotic expansion (5.21) for $E(\bar{X}^2)$, as $\lambda \to 0$.

Analogically, we can calculate the asymptotic expansions for the third and fourth moments of the ergodic distribution of the process $\bar{X}(t)$.

This completes the proof of Theorem 5.1. ■

Corollary 5.2. Let the conditions of Theorem 5.1 are satisfied. Then the following asymptotic expansion can be written for the variance of the ergodic distribution of the process $\bar{X}(t)$, as $\lambda \to 0$:

$$Var(\bar{X}) = \left[D_{31}(\alpha) - D_{21}^2(\alpha)\right]\lambda^2 + \left[B_{21}(\alpha) - 2B_{11}(\alpha)D_{12}(\alpha)\right]\lambda + \left[B_{22}(\alpha) - B_{11}^2(\alpha) - 2D_{21}(\alpha)B_{12}(\alpha)\right] + o(1).$$

Remark 5.1. Thus, we obtained the asymptotic expansions for the first four ergodic moments of the process $\bar{X}(t)$. Using these moments, it is possible to calculate skewness (γ_3) and kurtosis (γ_4) of the ergodic distribution of $\bar{X}(t)$:

$$\gamma_3 = \frac{E(X-a)^3}{\sigma^3}, \quad \gamma_4 = \frac{E(X-a)^4}{\sigma^4} - 3, \text{ where } a = E(X),\ \sigma^2 = Var(X).$$

Corollary 5.3. Under the conditions of Theorem 5.1, the following asymptotic expansions can be written for the skewness (γ_3) and kurtosis (γ_4) of the ergodic distribution of the process $\bar{X}(t)$, as $\lambda \to 0$:

$$\gamma_3 = \frac{D_{41}(\alpha) - 3D_{21}(\alpha)D_{31}(\alpha) + 2D_{21}^3(\alpha)}{[D_{31}(\alpha) - D_{21}^2(\alpha)]\sqrt{D_{31}(\alpha) - D_{21}^2(\alpha)}} + O(\lambda),$$

$$\gamma_4 = \frac{D_{51}(\alpha) + 6D_{31}(\alpha)D_{21}^2(\alpha) - 4D_{21}(\alpha)D_{41}(\alpha) - 3D_{21}^4(\alpha)}{[D_{31}(\alpha) - D_{21}^2(\alpha)]^2} - 3 + O(\lambda).$$

6. SIMULATION RESULTS

Thus, main aim of this study has been attained. But it is advisable to test an adequateness of approximate formulas to the exact ones. For this purpose, using the Monte Carlo experiments we can give the following simulation results. First, let's denote by $\hat{E}(X^n),\ (n = 2,3,4)$ and $\tilde{E}(X^n)$ the simulating and asymptotic values of the n^{th} moment ($E(X^n)$) of ergodic distribution of the process $X(t)$, respectively. Moreover, we put

$\Delta_n = \left|\hat{E}(X^n) - \tilde{E}(X^n)\right|$; $\delta_n = \dfrac{\Delta_n}{\hat{E}(X^n)}100\%$;

$Ap_n = 100\% - \delta_n$, n=1,2,3,4.

In other words, Δ_n, δ_n, Ap_n, (n=1,2,3,4) are the absolute error, relative error and accuracy percent between the simulating and asymptotic values of n^{th} ergodic moments of the process X(t), respectively. The following tables contain $\hat{E}(X^n),\ (n=1,2,3,4)$ computed by using Monte Carlo experiments when the variable η_1 has a Normal distribution with parameters (-1,1). For calculation of each quantity, 10^8 trajectories were simulated. Moreover, here s=1; $K \equiv E\theta_1 / E\xi_1 = 1$. Now, let's give the tables regarding the n^{th} moment ($E(X^n)$) of ergodic distribution of the process $X(t)$, when the variable η_1 has a Normal distribution with parameters (-1,1).

Table 1, $n=1$

λ	$\hat{E}(X)$	$\tilde{E}(X)$	Δ_1	$\delta_1(\%)$	$Ap_1(\%)$
10	06,16	05,61	0,55	8,97	91,02
20	11,80	11,26	0,53	4,50	95,49
30	17,39	16,90	0,48	2,81	97,18
40	23,05	22,53	0,52	2,26	97,73
50	28,70	28,16	0,54	1,90	98,09
60	33,69	33,79	0,09	0,28	99,71
70	39,40	39,41	0,01	0,04	99,95
80	45,05	45,04	0,01	0,03	99,96
90	50,65	50,66	0,01	0,02	99,97
100	56,30	56,29	0,01	0,01	99,98

Table 2, $n=2$

λ	$\hat{E}(X^2)$	$\tilde{E}(X^2)$	Δ_2	$\delta_2(\%)$	$Ap_2(\%)$
10	50,81	45,75	05,05	9,95	90,04
20	187,25	177,23	10,02	5,35	94,64
30	408,61	394,42	14,19	3,47	96,52
40	717,61	697,32	20,29	2,82	97,17
50	1112,18	1085,93	26,24	2,35	97,64
60	1590,06	1560,26	29,79	1,87	98,12
70	2132,74	2120,31	12,43	0,58	99,41
80	2770,62	2766,07	4,54	0,16	99,83
90	3494,75	3497,54	2,79	0,08	99,91
100	4317,05	4314,73	2,31	0,05	99,94

Table 3, $n=3$

λ	$\hat{E}(X^3)$	$\tilde{E}(X^3)$	Δ_3	$\delta_3(\%)$	$Ap_3(\%)$
10	482,17	433,40	48,77	10,11	89,88
20	3402,36	3230,80	171,56	5,04	94,95
30	11090,50	10642,20	448,30	4,04	95,95
40	25694,60	24917,60	777,00	3,02	96,97
50	49489,29	48307,00	1182,29	2,38	97,61
60	84664,72	83060,40	1604,32	1,89	98,10
70	133021,58	131427,80	1593,78	1,19	98,80
80	197272,14	195659,20	1612,94	0,81	99,18
90	279802,91	278004,60	1798,31	0,64	99,35
100	382294,86	380714,00	1580,86	0,41	99,58

Table 4, $n=4$

λ	$\hat{E}(X^4)$	$\tilde{E}(X^4)$	Δ_4	$\delta_4(\%)$	$Ap_4(\%)$
10	4906,11	4348	558,11	11,37	88,62
20	67344,24	63192	4152,24	6,16	93,83
30	324865,61	310032	14833,61	4,56	95,43
40	1004589,58	964768	39821,58	3,96	96,03
50	2398512,25	2333700	64812,25	2,70	97,29
60	4900069,25	4809528	90541,25	1,84	98,15
70	9005996,83	8871352	134644,83	1,49	98,50
80	15255401,81	15084672	170729,81	1,11	98,88
90	24328849,17	24101388	227461,17	0,93	99,06
100	36483926,18	36659800	175873,81	0,48	99,51

By using Monte Carlo experiments, it is shown that the given approximating formulas provide high accuracy even for small values of parameter λ. This indicates that the obtained formulas can safely be used for the various needs of the application.

7. CONCLUSIONS

In this study, the stationary characteristics of the process X(t) are investigated by using some asymptotic methods, whenever the random variable ζ_1, which describes credit policy, has ζ_1 has Pareto distribution with parameters (α,λ), as $\lambda \to 0$. To take the second and third terms in the asymptotic expansions, in addition to the first term, allow us to approximate the exact expressions for the moments of X(t) by some approximation formulas that they have sufficiently high accuracy. The evident and clear forms of the asymptotic expansions with three terms are allowed us to observe how the initial random variables ξ_1, η_1, ζ_1 influences the stationary characteristic's of the process. Therefore, this provides us to see that which parameters of the system influence the working of the insurance company and how does this happen. On the other hand, the first term of the obtained asymptotic expansions are depended on only the probabilistic characteristics of the random variable ζ_1 which describes the credit policy of insurance company. This knowledge demonstrates the dominant character of the credit policy. Thus, it is possible to keep under the control the whole working process of the company, by appropriately changing the properties of the credit policy.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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