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The vibration and stability of non-homogeneous orthotropic conical shells with clamped edges subjected to uniform external pressures

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ABSTRACT

In this paper an analytical procedure is given to study the free vibration and stability characteristics of homogeneous and non-homogeneous orthotropic truncated and complete conical shells with clamped edges under uniform external pressures. The non-homogeneous orthotropic material properties of conical shells vary continuously in the thickness direction. The governing equations according to the Donnell's theory are solved by Galerkin's method and critical hydrostatic and lateral pressures and fundamental natural frequencies have been found analytically. The appropriate formulas for homogeneous orthotropic and isotropic conical shells and for cylindrical shells made of homogeneous and non-homogeneous, orthotropic and isotropic materials are found as a special case. Several examples are presented to show the accuracy and efficiency of the formulation. The closed-form solutions are verified by accurate different solutions. Finally, the influences of the non-homogeneity, orthotropy and the variations of conical shells characteristics on the critical lateral and hydrostatic pressures and natural frequencies are investigated, when Young's moduli and density vary together and separately. The results obtained for homogeneous cases are compared with their counterparts in the literature.

1. Introduction

Conical shells represent one of the principal elements of aerospace and ship structures, pressure vessels and piping, for example, as reducers in piping, end closures for pressure vessels and liquid storage tanks, and roofs for tanks. The use of such shells as structural elements in various technological situations demands that the non-homogeneity of the materials should be taken into account for the analysis of the shell stability and vibration. Certain parts in aircraft and rockets have to operate under radiation and elevated temperatures and which cause non-homogeneity in the material, i.e., the elastic constants of the material become functions of space variables. Furthermore, the non-homogeneity of the materials stems from the effects of humidity, surface and thermal polishing processes and methods of production, which render the physical properties of materials, vary from point to point (random, piecewise continuous or continuous functions of coordinates). When non-homogeneous materials deform, they retain their shapes up to the point of rupture. Hence, in the computations of structural members made of such materials, the fundamental relations and governing equations of deformable body mechanics are applicable [1–3].

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Nomenclature

CC	complete cone
E_0	Young's modulus of the homogeneous isotropic material
E_{01} and E_{02}	Young's moduli of the homogeneous orthotropic material in S and θ directions, respectively
h	thickness of the conical shell
H	homogenous
f_{TC}	cyclic natural frequency (Hz) of truncated conical shells
f_{CC}	cyclic natural frequency (Hz) of complete conical shells
G_0	shear modulus of homogenous orthotropic material
L	length of the truncated cone
L_1	length of the cylindrical shell
$L_{ij};\quad i,j=1-4$	differential operators
$M_S, M_\theta, M_{S\theta}$	moment resultants
m	longitudinal wave number
n	circumferential wave number
$n_{HTC}, n_{LTC}, n_{fTC}$	circumferential wave numbers corresponding to P_{Hcr}^{TC}, P_{Lcr}^{TC}, f_{TC}, respectively
$n_{HCC}, n_{LCC}, n_{fCC}$	circumferential wave numbers corresponding to P_{Hcr}^{CC}, P_{Lcr}^{CC}, f_{CC}, respectively
NH	non-homogenous
P_{Hcr}^{TC}	critical uniform hydrostatic pressure of the truncated conical shell
P_{Hcr}^{CC}	critical uniform hydrostatic pressure of the complete conical shell
P_{Lcr}^{TC}	critical uniform lateral pressure of the truncated conical shell
P_{Lcr}^{CC}	critical uniform lateral pressure of the complete conical shell
$P_1 = P_2 = P_H$	uniform hydrostatic pressure
$P_1 = 0; P_2 = P_L$	uniform lateral pressure
R	radius of the cylindrical shell
R_1 and R_2	Radii of the small and large ends of the cone, respectively
S	axis lies on the curvilinear middle surface of the cone
S_1 and S_2	distances from the vertex to the small and large bases, respectively
TC	truncated cone
$T_S, T_\theta, T_{S\theta}$	force resultants
$T_s^0, T_\theta^0, T_{S\theta}^0$	membrane forces for the condition with zero initial moments
w	displacement of the middle surface in the normal direction
$\chi_S, \chi_{S\theta}, \chi_{S\theta}$	curvature components on the reference surface
$\varepsilon_S, \varepsilon_\theta, \varepsilon_{S\theta}$	strain components
$\varepsilon_S^0, \varepsilon_\theta^0, \varepsilon_{S\theta}^0$	strain components on the reference surface
$\varphi_1(\bar{\zeta}), \varphi_2(\bar{\zeta})$	non-homogeneity functions
γ	semi-vertex angle of the cone
λ	parameter
μ	coefficient of the non-homogeneity
v_0	Poisson's ratios of the isotropic material
v_{12}, v_{21}	Poisson's ratios of the orthotropic material
θ	axis is in the direction perpendicular to the $S-\zeta$ plane
ρ_0 and ρ_1	density of the homogeneous and non-homogenous materials, respectively
$\sigma_S, \sigma_\theta, \sigma_{S\theta}$	stress components
ω_{TC}	natural frequency (rad/s) of the circular truncated conical shell
$\bar{\omega}_{TC}$	natural frequency (rad/s) of the elliptical truncated conical shell
ω_{1TC}	dimensionless frequency parameter of the circular truncated conical shell
ω_{CC}	natural frequency (rad/s) of the circular complete conical shell
ω_{1CC}	dimensionless frequency parameter of the circular complete conical shell
$\bar{\omega}_{1CC}$	natural frequency (rad/s) of the elliptical complete conical shell
$\xi(t), \zeta\ (t)$	time dependent amplitudes
Ψ	stress function
ζ	thicknesses coordinate
x	parameter

In an up-to-date survey of literature, authors have come across various models to account for the material non-homogeneity proposed by researchers dealing with stability and vibration. Rao et al. [4] dealing with vibration of non-homogeneous isotropic thin plates have assumed linear variations for Young's modulus and density. Tomar et al. [5] have assumed exponential variations in the study of vibrational behavior of non-homogeneous isotropic plates. Heyliger and Juliani [6] have taken it to be a function of the radial coordinate in some vibration problems of the non-homogeneous isotropic shells. Erdogan and Wu [7] studied crack problems in FGM layers under thermal stresses which material properties obey the exponential distribution. Gutierrez et al. [8] gave solutions for the vibration frequencies of linear, parabolic and cubic variations of density using four approximate methods: optimized Rayleigh–Ritz, differential quadrature, finite elements, and lower-bound solution based on the stodola-vianello method. Zhang and Hasebe [9] have assumed the variation of the elasticity modulus to be unbounded and have used exponential functions of the radial coordinate in the elasticity solution for a non-homogeneous circular cylinder. Chakraverty and Petyt [10] have studied the vibration of non-homogeneous elliptic plates with radial tapering in Young's modulus and density with the constant Poisson ratio. The assumption of variation in which the parameter μ is same for Young's modulus as well as density does not seem to have any justification. In two significant contributions [11,12], Elishakoff has obtained unusual closed-form solutions for the axisymmetric vibrations of isotropic inhomogeneous circular plates and for the stability of isotropic inhomogeneous columns of assuming that inertial term/density and stiffness of the plate and columns are the polynomial functions of the radial and longitudinal coordinates, respectively. Gupta et al. [13], a more general model has been proposed in which, the Young's modulus and density are assumed to vary exponentially in radial direction in distinct manner. In all those studies the materials of the structural elements is isotropic.

Published literatures on analysis of composite orthotropic structures with variable material properties are limited in number. The rotation problem of a non-homogeneous orthotropic composite cylinder was considered by El-Naggar et al. [14]. A theoretical solution of a non-homogeneous orthotropic cylindrical shell is developed for the axisymmetric plane strain dynamic thermo-elastic problem and is usually solved using Laplace transform technique by Ding et al. [15]. Iesan and Quintanilla [16] have solved the Saint–Venant problem for inhomogeneous and orthotropic elastic cylinders where the constitutive coefficients are independent of the axial coordinate. Goldfeld [17] studied the influence of the variation of the stiffness coefficients on the buckling behavior and on the imperfection sensitivity of laminated conical shells.

The non-homogeneity of material properties across the thickness of plates and shells introduces additional difficulties and thus draws additional attention. Massalas et al. [18] have studied the dynamic instability of truncated conical shells under periodic compressive forces with the elasticity modulus as a linear function of thickness coordinate. Lee and Yu [19] derived a system of two-dimensional equations for vibrations of piezoelectric plates with thickness-graded material properties. Recently, Sofiyev and co-workers [20–23] a more general model has been proposed in which, the Young's moduli and density of the orthotropic materials of the shells are assumed to vary continuously and piecewise continuously in the thickness coordinate and have solved the static and dynamic stability problems of single-layer and laminated orthotropic cylindrical and conical shells with simple or freely supported edges.

Vibration and buckling of a general conical shell depend on boundary conditions as well as the geometric, material properties and loading conditions. Some closed-form solutions for the vibration and buckling of shells are available for a few types of boundary conditions. The literature on shells analysis particularly is full of the exact solutions for a shell simply supported at both ends. Approximate solutions are sought for other sets of boundary conditions like clamped–clamped, clamped-free, etc., by using numerical methods [24–35].

Publications about the closed-form solutions the stability and vibration of conical shells with clamped edges are limited in the literature, comparatively to the other types of edge restraints. Most of these works have been done for the vibration analyses of clamped cylindrical shells [36–45].

Furthermore, the excellent monographs on the vibration and stability of homogeneous shells by Volmir [46] and Leissa [47] contain one chapter devoted to conical shells and the references listed therein deal mostly with the study of isotropic conical shells.

Since, the free vibration and the stability of non-homogeneous orthotropic truncated and complete conical shells with clamped edges under various pressures have not been studied yet. In the present work, an attempt is made to address this problem. The conical shells are analyzed using the modified Donnell type stability and compatibility equations. Applying Galerkin methods to the foregoing equations, the buckling pressures and fundamental natural frequencies of homogeneous and non-homogeneous orthotropic conical shells with clamped edges are obtained. The Young's moduli and density of conical shells are defined as continuous functions of the thickness coordinate. Finally, the influences of the non-homogeneity, orthotropy and the variations of conical shells characteristics on critical lateral and hydrostatic pressures and fundamental natural frequencies are investigated, when Young's moduli and density vary together and separately.

2. Mathematical formulation of the problem

2.1. Kinematics

Consider a circular non-homogeneous orthotropic truncated conical shell as shown in Fig. 1, R_1 and R_2 indicate the radii of the cone at its small and large ends, γ denotes a semi-vertex angle of the cone, L is the cone length along its generator, h is the thickness. The reference surface of the conical shell is taken as the middle surface where an orthogonal coordinate system (S, θ, ζ) is fixed. The S-axis lies on the curvilinear middle surface of the cone, S_1 and S_2 being the coordinates of the points where this axis intersects the small and large bases, respectively. Furthermore, the ζ-axis is always normal to the moving

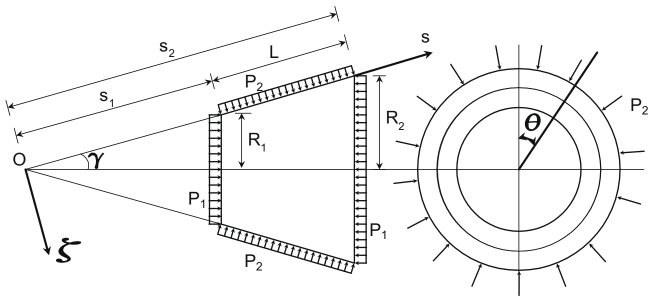

Fig. 1. Geometry of the conical shell.

S-axis, lying in the plane generated by the S-axis and the axis of the cone, and points inwards. The θ-axis is in the direction perpendicular to the $S-\zeta$ plane.

The axes of orthotropy are parallel to the curvilinear coordinates S and θ.

Based on the hypothesis of Kirchhoff–Love's first approximation which states that the strain components, $\varepsilon_S, \varepsilon_\theta$ and $\varepsilon_{S\theta}$, at any point of a conical shell can be expressed by a linear function of the normal coordinate ζ in terms of the reference strains and curvatures, each of these components may be expressed as [46]

$$\varepsilon_S = \varepsilon_S^0 + \zeta\chi_S, \quad \varepsilon_\theta = \varepsilon_\theta^0 + \zeta\chi_\theta, \quad \varepsilon_{S\theta} = \varepsilon_{S\theta}^0 + \zeta\chi_{S\theta}, \tag{1}$$

where $\varepsilon_S^0, \varepsilon_\theta^0, \varepsilon_{S\theta}^0$ and $\chi_S, \chi_{S\theta}, \chi_{S\theta}$ are, respectively, the strain and curvature components on the reference surface. They are defined by:

$$\chi_S = -\frac{\partial^2 w}{\partial S^2}, \quad \chi_\theta = -\frac{1}{S^2}\frac{\partial^2 w}{\partial \theta_1^2} - \frac{1}{S}\frac{\partial w}{\partial S}, \quad \chi_{S\theta} = -\frac{1}{S}\frac{\partial^2 w}{\partial S \partial \theta_1} + \frac{1}{S^2}\frac{\partial w}{\partial \theta_1} \tag{2}$$

in which $\theta_1 = \theta sin\gamma$, w is the displacement of the middle surface in the normal direction, positive towards the axis of the cone and assumed to be much smaller than the thickness.

2.2. Mathematical model of non-homogeneous orthotropic materials

The material of the conical shell is assumed to be orthotropic and non-homogeneous, such as Young's moduli $E_1(\bar{\zeta}), E_2(\bar{\zeta})$ and density $\rho(\bar{\zeta})$ must be described across the shell thickness [1–8,13,20–23]:

$$\lfloor [E_1(\bar{\zeta}), E_2(\bar{\zeta}), G(\bar{\zeta})] \rfloor = \bar{\varphi}_1(\bar{\zeta})[E_{01}, E_{02}, G_0]; \quad \rho(\bar{\zeta}) = \rho_0\bar{\varphi}_2(\bar{\zeta}), \quad \bar{\zeta} = \zeta/h, \tag{3}$$

where ζ is the thickness coordinate $-h/2 \leqslant \zeta \leqslant h/2$, E_{01} and E_{02} are the Young's moduli in the S and θ directions, respectively, G_0 is the shear modulus on the plane, and ρ_0 is the density of the homogeneous orthotropic material. Additionally,

$$\bar{\varphi}_j(\bar{\zeta}) = 1 + \mu\varphi_j(\bar{\zeta}), \quad j = 1, 2, \tag{4}$$

where $\varphi_1(\bar{\zeta})$ and $\varphi_2(\bar{\zeta})$ are continuous functions of non-homogeneity defining the variations of the Young's moduli and density, respectively, satisfying the condition $|\varphi_j(\bar{\zeta})| \leqslant 1$, and μ is a non-homogeneity coefficient, satisfying $0 \leqslant \mu < 1$. v_{12} and v_{21} are the Poisson's ratios, assumed to be constant and satisfying $v_{21}E_{01} = v_{12}E_{02}$ [48].

2.3. Constitutive equations

The stress–strain relation for thin non-homogeneous orthotropic truncated conical shells is:

$$\begin{pmatrix} \sigma_S \\ \sigma_\theta \\ \sigma_{S\theta} \end{pmatrix} = \begin{bmatrix} Q_{11}(\bar{\zeta}) & Q_{12}(\bar{\zeta}) & 0 \\ Q_{21}(\bar{\zeta}) & Q_{22}(\bar{\zeta}) & 0 \\ 0 & 0 & Q_{66}(\bar{\zeta}) \end{bmatrix} \begin{bmatrix} \varepsilon_S \\ \varepsilon_\theta \\ \varepsilon_{S\theta} \end{bmatrix}, \tag{5}$$

where

$$\begin{gathered} Q_{11}(\bar{\zeta}) = \frac{E_1(\bar{\zeta})}{1 - v_{12}v_{21}}, \quad Q_{22}(\bar{\zeta}) = \frac{E_2(\bar{\zeta})}{1 - v_{12}v_{21}}, \quad Q_{12}(\bar{\zeta}) = v_{21}Q_{11}(\bar{\zeta}), \\ Q_{21}(\bar{\zeta}) = v_{12}Q_{22}(\bar{\zeta}), \quad Q_{66}(\bar{\zeta}) = 2G(\bar{\zeta}) \end{gathered} \tag{6}$$

Moment resultants and in-surface force are defined as [46]:

$$[(T_S, T_\theta, T_{S\theta}), (M_S, M_\theta, M_{S\theta})] = \int_{-h/2}^{h/2} (1, \zeta)(\sigma_S, \sigma_\theta, \sigma_{S\theta}) d\zeta. \tag{7}$$

The relations between the forces T_S, T_θ and $T_{S\theta}$ and the stress function Ψ are given by

$$(T_S, T_\theta, T_{S\theta}) = \left(\frac{1}{S^2} \frac{\partial^2 \Psi}{\partial \theta_1^2} + \frac{1}{S} \frac{\partial \Psi}{\partial S}, \ \frac{\partial^2 \Psi}{\partial S^2}, \ -\frac{1}{S} \frac{\partial^2 \Psi}{\partial S \partial \theta_1} + \frac{1}{S^2} \frac{\partial \Psi}{\partial \theta_1} \right). \tag{8}$$

3. Stability and vibration equations

The orthotropic truncated conical shell subjected to uniform external pressures [22–27]:

$$T_S^0 = -0.5 P_1 S \tan\gamma, \quad T_\theta^0 = -P_2 S \tan\gamma, \quad T_{S\theta}^0 = 0, \tag{9}$$

where T_s^0, T_θ^0 and $T_{S\theta}^0$ are the membrane forces for the condition with zero initial moments.

When $P_1 = P_2 = P_H$, the external pressure turns into the uniform hydrostatic pressure. When $P_1 = 0;\ P_2 = P_L$, the external pressure turns into the uniform lateral pressure.

Substituting the Eq. (5) into the Eq. (7) then substituting the resulting expressions into the modified Donnell type stability and compatibility equations of truncated conical shells [46] together with relations (8) and (9), then considering new variable $S = S_2 e^x$, the governing equations in terms of w and Ψ are derived. They are a set of partial differential equations and their simplified expressions are given as follows:

$$\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} \Psi \\ w \end{pmatrix} = 0, \tag{10}$$

where $L_{ij};\ (i,j = 1\text{—}4)$ are the differential operators and the following definitions apply:

$$L_{11} = \delta_1 e^{-4x} \frac{\partial^4}{\partial x^4} + \delta_2 e^{-4x} \frac{\partial^3}{\partial x^3} + \delta_3 e^{-4x} \frac{\partial^2}{\partial x^2} + \delta_4 e^{-4x} \frac{\partial}{\partial x} - S_2 e^{-3x} \cot\gamma \frac{\partial}{\partial x} + S_2 e^{-3x} \cot\gamma \frac{\partial^2}{\partial x^2} + \delta_5 e^{-4x} \frac{\partial^4}{\partial \theta_1^4} + \delta_6 e^{-4x} \frac{\partial^4}{\partial x^2 \partial \theta_1^2}$$
$$+ \delta_7 e^{-4x} \frac{\partial^3}{\partial x \partial \theta_1^2} + \delta_8 e^{-4x} \frac{\partial^2}{\partial \theta_1^2}, \tag{11.1}$$

$$L_{12} = -\delta_9 e^{-4x} \frac{\partial^4}{\partial \theta_1^4} - \delta_{10} e^{-4x} \frac{\partial^4}{\partial x^2 \partial \theta_1^2} + \delta_{11} e^{-4x} \frac{\partial^3}{\partial x \partial \theta_1^2} - \delta_{12} e^{-4x} \frac{\partial^2}{\partial \theta_1^2} - \delta_{13} e^{-4x} \frac{\partial^4}{\partial x^4} + \delta_{14} e^{-4x} \frac{\partial^3}{\partial x^3} + \delta_{15} e^{-4x} \frac{\partial^2}{\partial x^2}$$
$$+ \delta_{16} e^{-4x} \frac{\partial}{\partial x} + S_2^3 e^{-x} (0.5 P_1 - P_2) \tan\gamma \frac{\partial}{\partial x} - 0.5 P_1 S_2^3 e^{-x} \tan\gamma \frac{\partial^2}{\partial x^2} - P_2 S_2^3 e^{-x} \tan\gamma \frac{\partial^2}{\partial \theta_1^2} - \rho_1 h S_2^4 \frac{\partial^2}{\partial t^2}, \tag{11.2}$$

$$L_{21} = \Delta_1 e^{-4x} \frac{\partial^4}{\partial \theta_1^4} + \Delta_2 e^{-4x} \frac{\partial^4}{\partial x^2 \partial \theta_1^2} - \Delta_3 e^{-4x} \frac{\partial^3}{\partial x \partial \theta_1^2} + \Delta_4 e^{-4x} \frac{\partial^2}{\partial \theta_1^2} + \Delta_5 e^{-4x} \frac{\partial^4}{\partial x^4}$$
$$+ \Delta_6 e^{-4x} \frac{\partial^3}{\partial x^3} + \Delta_7 e^{-4x} \frac{\partial^2}{\partial x^2} + \Delta_8 e^{-4x} \frac{\partial}{\partial x}, \tag{11.3}$$

$$L_{22} = -\Delta_9 e^{-4x} \frac{\partial^4}{\partial \theta_1^4} + \Delta_{10} e^{-4x} \frac{\partial^4}{\partial x^2 \partial \theta_1^2} + \Delta_{11} e^{-4x} \frac{\partial^3}{\partial x \partial \theta_1^2} + \Delta_{12} e^{-4x} \frac{\partial^2}{\partial \theta_1^2} - \Delta_{13} e^{-4x} \frac{\partial^4}{\partial x^4} + \Delta_{14} e^{-4x} \frac{\partial^3}{\partial x^3} + \Delta_{15} e^{-4x} \frac{\partial^2}{\partial x^2}$$
$$+ \Delta_{16} e^{-4x} \frac{\partial}{\partial x} + S_2 e^{-3x} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \right) \cot\gamma, \tag{11.4}$$

where t is time and the expressions $\delta_{\bar{k}}, \Delta_{\bar{k}} (\bar{k} = 1\text{—}16)$ and ρ_1 are given in Appendix A.

Eq. (10) are the basic equations for the free vibration (as the $P_1 = P_2 = 0$) and stability (neglecting the inertial term) of non-homogeneous orthotropic conical shells subjected to uniform external pressures.

Let us introduce functions w and Ψ defined by the relations [37]

$$w = e^{\lambda x} w_1(x,t) \cos(\beta_2 \theta_1), \quad \Psi = S_2 e^{(\lambda+1)x} \Psi_1(x,t) \cos(\beta_2 \theta_1), \tag{12}$$

where λ is a parameter which is found from minimum conditions of critical stresses and frequencies and the following definitions apply:

$$\beta_2 = \frac{n}{\sin\gamma}; \quad x_0 = \ln \frac{S_1}{S_2}. \tag{13}$$

Multiplying the first equation of the system (10) by $wS_2^2e^{2x}d\theta_1 dx$ and the second equation of the system (10) by $\Psi S_2^2e^{2x}d\theta_1 dx$, for $-x_0 \leqslant x \leqslant 0$ and $0 \leqslant \theta_1 \leqslant 2\pi sin\gamma$, then considering Eq. (12) and applying Galerkin's method to Eq. (10), after integrating with coordinate θ_1, the following differential equations depending on the parameter x are obtained:

$$\int_{-x_0}^{0}\left\{\begin{array}{l}\{\delta_1[e^{(\lambda+1)x}\Psi_1]_{,xxxx}+\delta_2[e^{(\lambda+1)x}\Psi_1]_{,xxx}\}S_2e^{(\lambda-2)x}w_1\\ \quad+(\delta_3-\delta_6\beta_2^2)S_2e^{(\lambda-2)x}w_1[e^{(\lambda+1)x}\Psi_1]_{,xx}+(\delta_4-\delta_7\beta_2^2)S_2e^{(\lambda-2)x}w_1[e^{(\lambda+1)x}\Psi_1]_{,x}\\ \quad+S_2^2e^{(\lambda-1)x}w_1\{[e^{(\lambda+1)x}\Psi_1]_{,xx}-[e^{(\lambda+1)x}\Psi_1]_{,x}\}\cot\gamma\\ \quad+(\delta_5\beta_2^2-\delta_8)\beta_2^2S_2e^{(2\lambda-1)x}w_1\Psi_1+[\delta_{12}-\delta_9\beta_2^2e^{4x}]e^{(2\lambda-2)x}\beta_2^2w_1^2\\ \quad+[\delta_{10}(e^{\lambda x}w_1)_{,xx}-\delta_{11}(e^{\lambda x}w_1)_{,x}]\beta_2^2e^{(\lambda-2)x}w_1\\ \quad+[\delta_{14}(e^{\lambda x}w_1)_{,xxx}-\delta_{13}(e^{\lambda x}w_1)_{,xxxx}+\delta_{15}(e^{\lambda x}w_1)_{,xx}+\delta_{16}(e^{\lambda x}w_1)_{,x}]e^{(\lambda-2)x}w_1\\ \quad+(0.5P_1-P_2)S_2^3e^{(\lambda+1)x}w_1(e^{\lambda x}w_1)_{,x}\tan\gamma-0.5P_1S_2^3e^{(\lambda+1)x}w_1(e^{\lambda x}w_1)_{,xx}\tan\gamma\\ \quad+P_2S_2^3\beta_2^2e^{(2\lambda+1)x}w_1^2\tan\gamma-\rho_1hS_2^4e^{(2\lambda+2)x}w_1w_{1,tt}\end{array}\right\}dx=0, \tag{14}$$

$$\int_{-x_0}^{0}\left\{\begin{array}{l}(\Delta_1\beta_2^2-\Delta_4)\beta_2^2S_2e^{2\lambda x}\Psi_1^2+(\Delta_7-\Delta_2\beta_2^2)S_2e^{(\lambda-1)x}\Psi_1[e^{(\lambda+1)x}\Psi_1]_{,xx}\\ \quad+(\Delta_8-\Delta_3\beta_2^2)S_2e^{(\lambda-1)x}\Psi_1[e^{(\lambda+1)x}\Psi_1]_{,x}\\ \quad+\{\Delta_5[e^{(\lambda+1)x}\Psi_1]_{,xxxx}+\Delta_6[e^{(\lambda+1)x}\Psi_1]_{,xxx}\}S_2e^{(\lambda-1)x}\Psi_1\\ \quad-(\Delta_9\beta_2^2+\Delta_{12})\beta_2^2e^{(2\lambda-1)x}\Psi_1w_1+(\Delta_{15}-\Delta_{10}\beta_2^2)e^{(\lambda-1)x}\Psi_1(e^{\lambda x}w_1)_{,xx}\\ \quad+(\Delta_{16}-\Delta_{11}\beta_2^2)e^{(\lambda-1)x}\Psi_1(e^{\lambda x}w_1)_{,x}\\ \quad+[\Delta_{14}(e^{\lambda x}w_1)_{,xxx}-\Delta_{13}(e^{\lambda x}w_1)_{,xxxx}]e^{(\lambda-1)x}\Psi_1\\ \quad+[(e^{\lambda x}w_1)_{,xx}-(e^{\lambda x}w_1)_{,x}]S_2e^{\lambda x}\Psi_1\cot\gamma\end{array}\right\}dx=0, \tag{15}$$

where $(\bullet)_{,x\ldots}$ denote differentiations with respect to the parameter x.

Eqs. (14) and (15) can be integrated for different boundary conditions.

4. The solution of the eigenvalue problem

For the present truncated, circular, non-homogenous orthotropic conical shell, the clamped boundary conditions at both ends are considered and expressed as [25,37,46]

$$w=\frac{\partial w}{\partial S}=0;\quad T_S=0;\quad T_{S\theta}=0\quad \text{at } S=S_1\quad\text{and}\quad S=S_2. \tag{16}$$

The approximate functions field may be taken as [37]

$$w_1=\xi(t)\sin^2(\beta_1x),\quad \Psi_1=\zeta(t)\sin^2(\beta_1x), \tag{17}$$

where $\xi(t)$ and $\zeta(t)$ are time dependent amplitudes and $\beta_1=m\pi x_0$.

When $P_1 = P_2 = 0$, substituting Eq. (17) into Eqs. (14) and (15), after integration according to x and some manipulations, for the natural frequency ω_{TC} (rad/s) of free vibration of the non-homogeneous orthotropic truncated conical shells with clamped edges, the following expression is obtained:

$$\omega_{TC}=\sqrt{\frac{(Q_1\eta_{-1}+Q_8)(Q_6\eta_{-1}+Q_8)}{Q_5Q_7\eta_2}+\frac{Q_2\eta_{-2}}{Q_7\eta_2}}, \tag{18}$$

where the following definitions apply:

$$\begin{aligned}Q_1&=8\delta_1[16\beta_1^4+(16\lambda^2-16\lambda-50)\beta_1^2+24\lambda^4-48\lambda^3-72\lambda^2+96\lambda+96]\\&\quad-36\delta_2(4\beta_1^2+\lambda^2-\lambda-2)+4(\delta_6\beta_2^2-\delta_3)(2\lambda^2-2\lambda-13+8\beta_1^2)\\&\quad+36\delta_4+24\delta_5\beta_2^4-12(3\delta_7+2\delta_8)\beta_2^2;\\Q_2&=8\{3\delta_9\beta_2^4+3(\delta_{11}-\delta_{12})\beta_2^2+(\delta_{15}+\delta_{10}\beta_2^2)(\lambda^2-2\lambda-2+4\beta_1^2)\\&\quad+\delta_{13}[16\beta_1^4+8(2\lambda^2-4\lambda-1)\beta_1^2+3\lambda^4-12\lambda^3+12\lambda^2]\\&\quad+3\delta_{14}(4\beta_1^2+\lambda^2-2\lambda)-3\delta_{16}\};\\Q_3&=4(\lambda^2+\lambda-2+4\beta_1^2)S_2^3\tan\gamma;\quad Q_4=12(2\beta_2^2+1)S_2^3\tan\gamma;\\Q_5&=8\{\Delta_5[16\beta_1^4+8(2\lambda^2-3)\beta_1^2+3\lambda^4-6\lambda^2+3]+3\Delta_8+3\Delta_1\beta_2^4\\&\quad+3(\Delta_3-\Delta_4)\beta_2^2-3\Delta_6(4\beta_1^2+\lambda^2-1)+(\Delta_2\beta_2^2-\Delta_7)(\lambda^2+4\beta-3)\};\end{aligned}$$

$$Q_6 = -24\Delta_9\beta_2^4 - 12(\Delta_{11} + 2\Delta_{12})\beta_2^2 + 12\Delta_{16} + 8\Delta_{10}\beta_2^2(\lambda^2 + 4\beta_1^2 - 3) + 8\Delta_{14}[16\beta_1^4 + 2(8\lambda^2 - 8\lambda - 1)\beta_1^2 + 3\lambda^4 - 6\lambda^3 + 3\lambda^2] + 4\Delta_{13}(2\lambda^2 - 2\lambda - 1 + 8\beta_1^2) - 12\Delta_{15}(4\beta_1^2 + \lambda^2 - \lambda);$$

$$Q_7 = 24\rho_1 hS_2^4; \quad Q_8 = -(8\lambda^2 + 32\beta_1^2)S_2 \cot\gamma,$$

$$\eta_i = \frac{1 - e^{-(2\lambda+i)x_0}}{1 - e^{-2\lambda x_0}} \frac{\lambda[(\lambda^2 + \beta_1^2)][(\lambda^2 + 4\beta_1^2)]}{(\lambda + 0.5i)\left[(\lambda + 0.5i)^2 + \beta_1^2\right]\left[(\lambda + 0.5i)^2 + 4\beta_1^2\right]}; \quad i = -2, -1, 1, 2 \tag{19}$$

The cyclic natural frequency f_{TC} (Hz) for the non-homogeneous orthotropic truncated conical shells with clamped edges is defined as

$$f_{TC} = \omega_{TC}/2\pi. \tag{20}$$

The dimensionless frequency parameter ω_{1TC} for the non-homogeneous orthotropic truncated conical shells with clamped edges is defined as

$$\omega_{1TC} = \omega_{TC}R_2\sqrt{(1 - \nu_{12}\nu_{21})\rho_0/(E_{01}E_{02})^{0.5}}. \tag{21}$$

At statically case, substituting Eq. (17) into the Eqs. (14) and (15), after integration according to x and some manipulations, for the critical uniform hydrostatic pressure ($P_1 = P_2 = P_H$) of the non-homogeneous orthotropic truncated conical shells with clamped edges, the following expression is obtained:

$$P_{Hcr}^{TC} = \frac{(Q_1\eta_{-1} + Q_8)(Q_6\eta_{-1} + Q_8) + Q_5Q_2\eta_{-2}}{Q_5(Q_3 + Q_4)\eta_1}. \tag{22}$$

For the critical uniform lateral pressure ($P_1 = 0; P_2 = P_L$) of the non-homogeneous orthotropic truncated conical shells with clamped edges, the following expression is obtained:

$$P_{Lcr}^{TC} = \frac{(Q_1\eta_{-1} + Q_8)(Q_6\eta_{-1} + Q_8) + Q_5Q_2\eta_{-2}}{Q_5Q_4\eta_1}. \tag{23}$$

The minimum values of critical parameters of the non-homogeneous orthotropic truncated conical shells are obtained by minimizing Eqs. (18), (20)–(23) with respect to m, n and λ.

The present Eqs. (18), (20)–(23) can be used also for the study of non-homogeneous orthotropic complete circular conical shells with clamped edges:

(a) The truncated conical shell is transformed into the complete conical shell when $R_1 \to 0$. In this case, $P_{Hcr}^{TC}; P_{Lcr}^{TC}; \omega_{TC}; f_{TC}; \omega_{1TC}$ in Eqs (18)–(23) are transformed into $P_{Hcr}^{CC}; P_{Lcr}^{CC}; \omega_{CC}; f_{CC}; \omega_{1CC}$, respectively.
(b) The truncated conical shell is transformed into the cylindrical shell when $\gamma \to 0$. If $\gamma = \pi/180000 \to 0$ are substituted in Eqs. (18)–(23) corresponding formulas for clamped cylindrical shells are obtained. In this case, $P_{Hcr}^{TC}; P_{Lcr}^{TC}; \omega_{TC}; f_{TC}; \omega_{1TC}$ in Eqs. (18), (20)–(23) are transformed into $P_{Hcr}^{cyl}; P_{Lcr}^{cyl}; \omega_{cyl}; f_{cyl}; \omega_{1cyl}$, respectively.

After the various numerical computations and analyses for critical parameters of conical and cylindrical shells, the following generalized values are obtained for the parameter λ (see, also Ref. [37]):

(a) The minimum values of hydrostatic and lateral buckling pressures of the clamped truncated conical shell are obtained approximately at $\lambda = 4$.
(b) The values of the fundamental cyclic frequency and dimensionless fundamental frequency parameter of the clamped truncated conical shell are obtained approximately,

$\lambda = 2.4$ for $x_0 < 1.6$;
$\lambda = 2.8$ for $1.6 \leqslant x_0 \leqslant 2.5$;
$\lambda = 3.2$ for $x_0 > 2.5$.

(c) The minimum values of hydrostatic and lateral buckling pressures for the clamped complete conical shell are getting approximately at $\lambda = 5.7$.
(d) The values of fundamental cyclic frequency and dimensionless fundamental frequency parameter for the clamped complete conical shell are getting approximately at $\lambda = 3.68$.
(e) The values of critical parameters for clamped cylindrical shells are getting at $\lambda = 0$.

Furthermore the longitudinal wave number m is equal to one for the cylindrical and conical shells subjected to uniform external pressures. In numerical computations part, by taking into account these values for the parameter λ and for the longitudinal wave number $m = 1$, critical parameters are minimized only according to n.

5. Numerical results and discussion

5.1. Comparative studies

In order to validate the proficiency of the present study, several numerical examples are carried out for comparisons. The fundamental frequency parameters of homogeneous isotropic cylindrical shells with clamped edges are presented in Table 1 and Table 2. By taking $\gamma \to 0$; $R_2 = R_1 = R; L = L_1$; $\mu = 0$; $E_{01} = E_{02} = E_0$; $\nu_{12} = \nu_{21} = \nu_0$ into the present formulations; and then the non-homogeneous orthotropic conical shell becomes a homogeneous isotropic cylindrical shell. Here R and L_1 are the radius and length of cylindrical shells, respectively.

The comparisons of the fundamental frequency parameter $\omega^*_{1\mathrm{cyl}}$ and the fundamental mode number n^* of the same homogeneous isotropic cylindrical shells to those in [36,40,45] are presented in Table 1. Present results are little higher than results of [36,40] and lower than results of [45]. This may be due to El-Mously [45] used Flugge theory. Furthermore, present results are changing between results of [36,40] and [45]. This shows our results are reliable.

Secondly, in Table 2, the fundamental frequency parameter $\omega_{1\mathrm{cyl}}$ of a homogeneous isotropic cylindrical shell are calculated and compared to those obtained by Loy et al. [42] and Zhang et al. [44]. Close agreements between the results of this paper and those reported in Refs. [42,44] is observed from Table 2.

The third and fourth comparisons, as shown in Tables 3 and 4, are for the conical shells. By taking $\mu = 0$, $E_{01} = E_{02} = E_0$, $\nu_{12} = \nu_{21} = \nu_0$ into the present formulations; and then the laminated non-homogeneous orthotropic conical shell becomes a homogeneous isotropic conical shell.

The values of the dimensionless critical hydrostatic pressure of homogeneous isotropic truncated conical shells are formed for RF1 (or RF3) clamped boundary conditions, i.e. for $w = \frac{\partial w}{\partial S} = 0; T_S = 0; T_{S\theta} = 0$ (or $v = \frac{\partial w}{\partial S} = 0; T_S = 0; T_{S\theta} = 0$) at $S = S_1$ and $S = S_2$ of Table 3 of Singer et al. [25] and compared with present results (see Table 3). The values of the

Table 1
Comparison of values of the fundamental frequency parameter $\omega^*_{1\mathrm{cyl}} = \omega_{\mathrm{cyl}}(R^2/h)\sqrt{(1-\nu_0^2)\rho_0/E_0}$ and the fundamental mode number n^* (given in parentheses) for clamped cylindrical shells.

Comparative studies	L_1/R	$\omega^*_{1\mathrm{cyl}}$; (n^*)	
		R/h = 100	R/h = 200
Weingarten [36]	1	23.2003(7)	35.6664(9)
Koga [40]		22.5008(7)	34.8558(9)
El-Mously [45]-Flugge theory		29.1461(7)	42.6632(9)
Present study		26.9334(7)	38.1112(10)
Weingarten [36]	2	13.0893(6)	19.6931(7)
Koga [40]		12.8748(6)	19.7827(7)
El-Mously [45]-Flugge theory		15.3271(6)	22.4040(7)
Present study		13.7215(5)	19.3016(7)

Table 2
Comparison of the fundamental frequency parameter value $\omega_{1\mathrm{cyl}}$ for a homogeneous isotropic cylindrical shell with clamped edges (L_1/R = 20, h/R = 0.01).

$\omega_{1\mathrm{cyl}} = \omega_{\mathrm{cyl}}R\sqrt{(1-\nu_0^2)\rho_0/E_0}$		
Loy et al. [42]	Zhang et al. [44]	Present study
0.01393(2)	0.01405(2)	0.01401(2)

Table 3
Dimensionless critical hydrostatic pressures of homogeneous isotropic truncated conical shells with clamped edges compared to those given by Singer et al. [25] for the different semi-vertex angle γ and L/R_1 ratio (R_1/h = 100).

$\left(P^{\mathrm{TC}}_{\mathrm{Hcr}}/E_0\right) \times 10^6$						
	Singer et al. [25]	Present study	Singer et al. [25]	Present study	Singer et al. [25]	Present study
γ	L/R_1 = 0.5		L/R_1 = 1		L/R_1 = 2	
10°	24.61(12)	25.351(12)	9.645(9)	10.202(9)	4.122(7)	4.2980(7)
30°	18.97(12)	19.441(12)	6.695(10)	6.9532(10)	2.436(8)	2.5213(8)
50°	12.24(12)	12.502(12)	3.869(10)	3.9711(10)	1.270(8)	1.2701(9)
70°	5.648(10)	5.7645(10)	1.528(8)	1.557(8)	0.4432(8)	0.4440(8)

*Numbers in brackets indicate the number of circumferential waves.

Table 4
Comparison of dimensionless frequency parameters for homogeneous isotropic circular truncated and complete conical shells with clamped edges (E_0 = 1.93 × 10^{11} Pa; ν_0 = 0.3; ρ_0 = 8000 kg/m^3; h = 0.001 m; R_1 = 0.1 m and $R_1 = 10^{-50}$ m; R_2 = 0.175 m; L = 0.6 m).

n	$\omega_{1TC} = \omega_{TC}R_2\sqrt{(1-\nu_0^2)\rho_0/E_0}$ (λ = 2.4)		$\omega_{1CC} = \omega_{CC}R_2\sqrt{(1-\nu_0^2)\rho_0/E_0}$ (λ = 3.68)	
	Present study	Ref. [37]	Present study	Ref. [37]
1	0.4199155633	0.4199155629	0.3692448620	0.3692448624
2	0.1736865575	0.1736865573	0.1463205658	0.1463205660
3	0.0892420257	0.0892420256	0.0913505680	0.0913505688
4	0.0672561201	0.0672561200	0.1141245329	0.1141245329
5	0.0769219418	0.0769219418	0.1688925636	0.1688925635
6	0.1022555996	0.1022555996	0.2407318617	0.2407318617
7	0.1361029256	0.1361029257	0.3267351685	0.3267351685
8	0.1763352747	0.1763352749	0.4262799402	0.4262799402
9	0.2223353759	0.2223353761	0.5392017764	0.5392017764
10	0.2739033361	0.2739033363	0.6654489578	0.6654489574

dimensionless critical hydrostatic pressure of homogeneous isotropic truncated conical shells with RF1 and RF3 clamped edges are nearly same. It can be seen that the present results are in good agreement with results of Singer et al. [25].

Comparisons of fundamental frequency parameters ω_{1TC} and ω_{1CC} of homogeneous isotropic circular truncated and complete conical shells, respectively, with results presented in Ref. [37] are given in Table 4. The expression of the natural frequency $\bar{\omega}_{TC}$ for the homogeneous isotropic elliptical truncated conical shell with clamed edges that obtained in Ref. [37] is given in Appendix B. When $\varepsilon = 0$, fundamental frequency parameters $\bar{\omega}_{1TC}$ and $\bar{\omega}_{1CC}$ for elliptical truncated and complete conical shells are transformed into fundamental frequency parameters ω_{1TC} and ω_{1CC} of circular truncated and complete conical shells, respectively (see Appendix B). An excellent agreement between the present and the reference results is observed.

5.2. Vibration and buckling analyses

Numerical computations, for homogeneous and non-homogeneous orthotropic truncated and complete conical shells with clamped edges have been carried out using expressions (18), (20)–(23). The results are presented in Figs. 2–8 and Tables 5 and 6.

Homogeneous and non-homogeneous orthotropic conical shells with different types of geometry are considered and their critical dimensionless lateral and hydrostatic pressures and fundamental cyclic natural frequencies computed. The non-homogeneity functions of the materials of conical shells are assumed to be power and exponential functions [1,3,8,20–24] which $\varphi_j(\bar{\zeta}) = \bar{\zeta};\ \bar{\zeta}^2;\ e^{-0.1|\bar{\zeta}|}\cos(0.5\bar{\zeta});\ j = 1, 2$.

Composite material properties are given below [48]:

$$E_{01} = 1.724 \times 1011\ \mathrm{N/m^2}, \quad E_{02} = 7.79 \times 109\ \mathrm{N/m^2}, \quad \nu_{21} = 0.35, \quad \rho_0 = 1530\ \mathrm{kg/m^3}.$$

In tables and figures, P_{Hcr}^{TC} (MPa); P_{Lcr}^{TC} (MPa), f_{TC} (Hz) and $n_{HTC}, n_{LTC}, n_{fTC}$ are hydrostatic buckling pressure, lateral buckling pressure, fundamental cyclic frequency and corresponding circumferential wave numbers, respectively, for a truncated conical shell. P_{Hcr}^{CC} (MPa), P_{Lcr}^{CC} (MPa), f_{CC} (Hz) and $n_{HCC}, n_{LCC}, n_{fCC}$ are hydrostatic buckling pressure, lateral buckling pressure, fundamental cyclic frequency and corresponding circumferential wave numbers, respectively, for the complete conical shell.

The values of buckling hydrostatic and lateral pressures, fundamental cyclic frequencies and corresponding wave numbers for homogeneous and non-homogeneous orthotropic truncated and complete conical shells are listed in Table 5 with respect to the semi-vertex angle γ. Furthermore, Figs. 2–4 are formed, by using values of critical hydrostatic and lateral pressures, and fundamental cyclic frequencies for homogeneous and non-homogeneous orthotropic truncated and complete conical shells versus semi-vertex angle, γ, that are given in Table 5. In Figs. 2–4 and Table 5, truncated and complete conical shells have geometrical parameters as R_1 = 1 m, R_2 = 2 m, h = 0.01 m and $R_1 = 10^{-50} \approx 0$, R_2 = 2 m, h = 0.01 m, respectively.

The values of P_{Hcr}^{TC} (MPa) and P_{Lcr}^{TC} (MPa) increase as $10^\circ \leqslant \gamma \leqslant 50^\circ$ and decrease as $50^\circ < \gamma \leqslant 70^\circ$ and the values of P_{Hcr}^{CC} (MPa) and P_{Lcr}^{CC} (MPa) increase as $10^\circ \leqslant \gamma \leqslant 45^\circ$ and decrease as $45^\circ < \gamma \leqslant 70^\circ$, whereas, corresponding wave numbers increase for homogeneous and non-homogeneous orthotropic truncated and complete conical shells, as the semi-vertex angle γ is increased. As the semi-vertex angle, γ, is increased, the minimum values of f_{TC} (Hz) continuously increase, whereas, corresponding wave numbers increase as $10^\circ \leqslant \gamma \leqslant 30^\circ$ and decrease as $30^\circ < \gamma \leqslant 70^\circ$ for homogeneous and non-homogeneous orthotropic truncated conical shells. The values of f_{CC} (Hz) increase as $10^\circ \leqslant \gamma \leqslant 45^\circ$ and decrease as $45^\circ < \gamma \leqslant 70^\circ$, whereas, corresponding wave numbers increase as $10^\circ \leqslant \gamma \leqslant 40^\circ$ and decrease as $40^\circ < \gamma \leqslant 70^\circ$ for homogeneous and non-homogeneous orthotropic complete conical shells.

Fig. 2 shows variations of the values of (a) f_{TC} (Hz) and (b) f_{CC} (Hz) for H and NH orthotropic conical shells versus the semi-vertex angle, γ, for different non-homogeneity functions. It is observed that the values of the fundamental cyclic frequencies of complete conical shells are lower than truncated conical shells.

Fig. 3 shows variations of the values of hydrostatic buckling pressures for the homogeneous and non-homogeneous, orthotropic truncated and complete conical shells versus the semi-vertex angle, γ, for the parabolic non-homogeneity

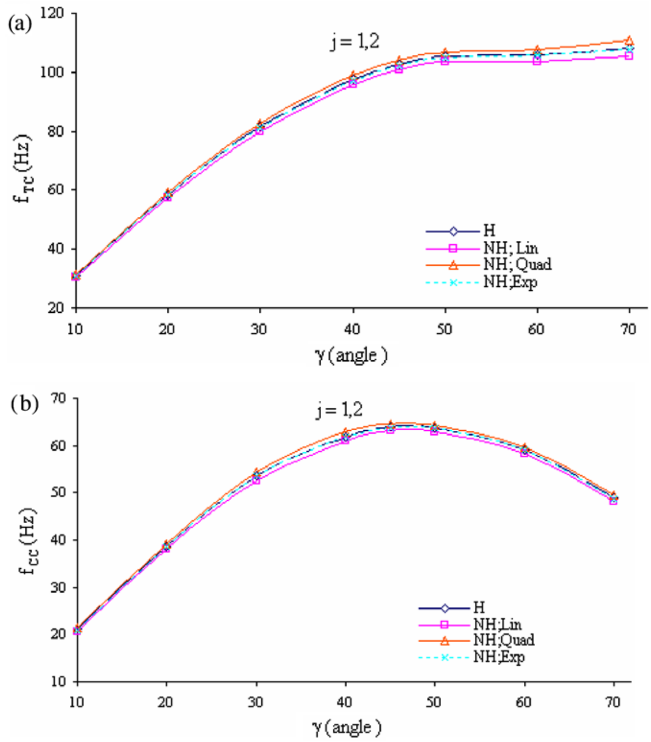

Fig. 2. Variations of the values of (a) f_{TC}(Hz) and (b) f_{CC}(Hz) for H and NH orthotropic conical shells versus the semi-vertex angle γ (R_1 = 1 m; R_2 = 2 m; h = 0.01 m; μ = 0.9; j = 1, 2).

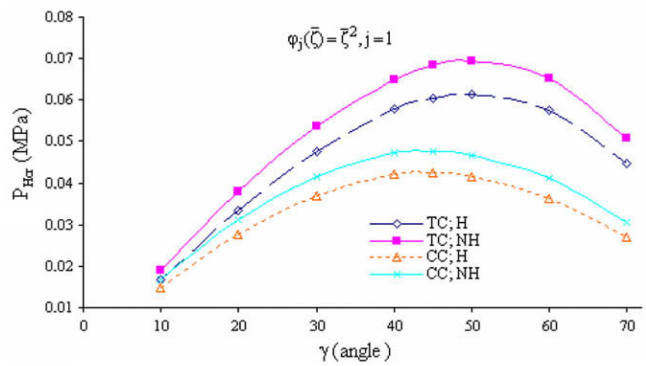

Fig. 3. Variations of the values of the hydrostatic buckling pressure for H and NH, orthotropic conical shells versus the semi-vertex angle, γ.

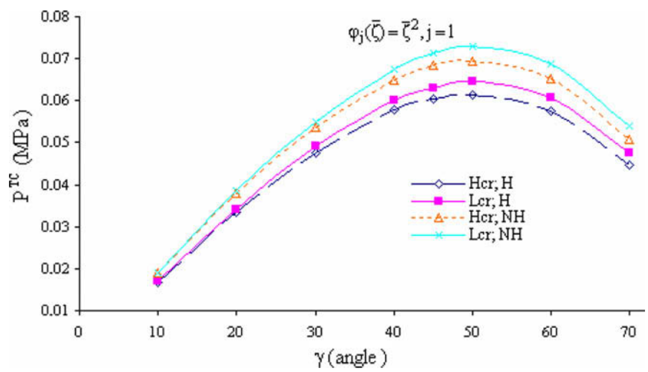

Fig. 4. Variations of the values of P_{Hcr}^{TC} (MPa) and P_{Lcr}^{TC} (MPa) for H and NH, orthotropic truncated conical shells versus the semi-vertex angle, γ.

function. It is observed that the values of the hydrostatic buckling pressure of a complete conical shell are lower than the truncated conical shell and the effect of the parabolic variation of the Young' moduli on the hydrostatic buckling pressure are important (see, also Table 5).

Fig. 4 shows variations of the values of hydrostatic and lateral buckling pressures for homogeneous and non-homogeneous, orthotropic truncated conical shells versus the semi-vertex angle, γ, for the parabolic non-homogeneity function. The values of the hydrostatic buckling pressure are lower than the values of the lateral buckling pressure for a truncated conical shell. Furthermore, the effect of the parabolic variation of the Young' moduli on the buckling pressures is considerable.

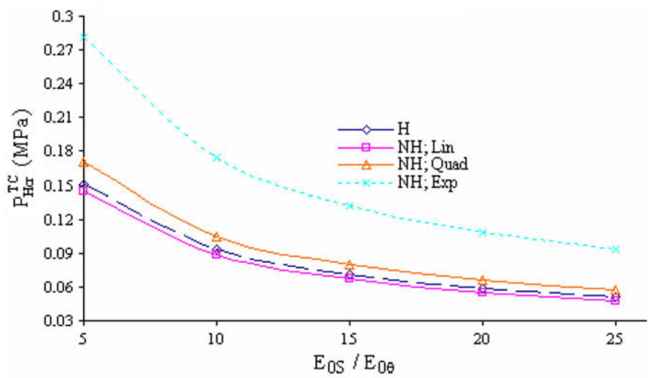

Fig. 5. Variations of the values of P_{Hcr}^{TC} (MPa) for H and NH orthotropic truncated conical shells versus the ratio $E_{0S}/E_{0\theta}$ ($L/R_1 = 2$; $\gamma = 30°$; $R_2/h = 200$; $\lambda = 4$).

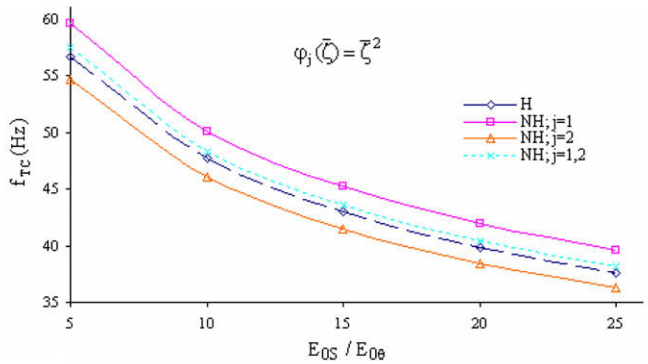

Fig. 6. Variations of the values of f_{TC} (Hz) for H and NH orthotropic truncated conical shells versus the ratio $E_{0S}/E_{0\theta}$ ($L/R_1 = 2$; $\gamma = 30°$; $R_2/h = 200$; $\lambda = 4$).

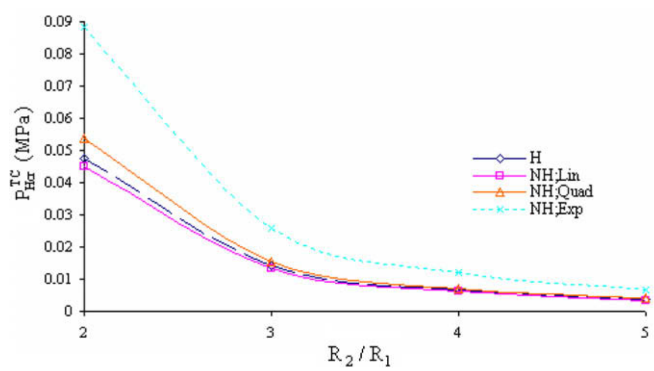

Fig. 7. Variations of the values of P_{Hcr}^{TC} (MPa) for H and NH orthotropic conical shells versus the ratio R_2/R_1 for various non-homogeneity functions ($h = 0.01$ m; $\gamma = 30°$; $\mu = 0.9$).

In Table 6 variations of the values of $P_{\mathrm{Hcr}}^{\mathrm{TC}}$ (MPa), $P_{\mathrm{Lcr}}^{\mathrm{TC}}$ (MPa), f_{TC} (Hz) and corresponding circumferential wave numbers n_{HTC}, n_{LTC}, n_{fTC}, respectively for the homogeneous and non-homogeneous orthotropic truncated conical shells with different non-homogeneity cases, versus the ratio L/R_1 are presented. As expected, as the dimensionless length parameter L/R_1 increases, the values of the dimensionless hydrostatic and lateral buckling pressures and corresponding wave numbers decrease for the homogeneous and non-homogeneous orthotropic truncated conical shells with all the non-homogeneity cases. Also, as the dimensionless length parameter L/R_1 increases, the values of fundamental cyclic frequencies decreases, whereas, corresponding wave numbers increase. It is observed that the effect of the non-homogeneity on the values of f_{TC} (Hz) is little, in the all cases. As the ratio L/R_1 increases, the percentage effects of the non-homogeneity on the values

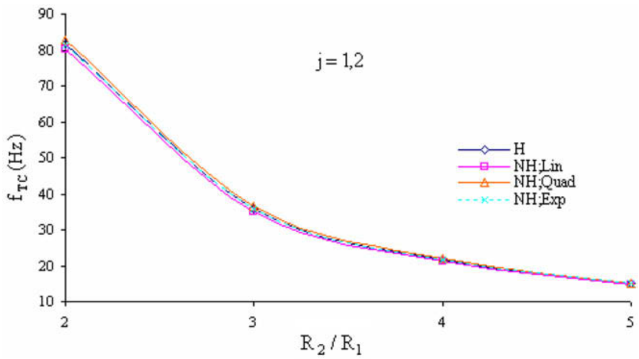

Fig. 8. Variations of the values of f_{TC} (Hz) for H and NH orthotropic conical shells versus the ratio R_2/R_1 for various non-homogeneity functions ($h = 0.01$ m; $\gamma = 30^{\circ}$; $\mu = 0.9$; $j = 1, 2$).

Table 5
Variations of the values of fundamental cyclic frequencies, buckling pressures and corresponding circumferential wave numbers for H and NH, orthotropic conical shells versus the semi-vertex angle γ for different non-homogeneity functions ($R_2/h = 200$; $\mu = 0.9$; $j = 1, 2$).

γ	$\varphi_j(\bar{\zeta})$; $j = 1, 2$	$P_{\mathrm{Hcr}}^{\mathrm{TC}}$ (MPa); (n_{cr}, $\lambda = 4$)	$P_{\mathrm{Lcr}}^{\mathrm{TC}}$ (MPa); (n_{cr}, $\lambda = 4$)	$P_{\mathrm{Hcr}}^{\mathrm{CC}}$ (MPa); (n_{cr}, $\lambda = 5.7$)	$P_{\mathrm{Lcr}}^{\mathrm{CC}}$ (MPa); (n_{cr}, $\lambda = 5.7$)	f_{TC} (Hz); (n_{cr}, $\lambda = 2.4$)	f_{CC} (Hz); (n_{cr}, $\lambda = 3.68$)
10°	Hom.	0.0169(8)	0.0170(8)	0.0148(6)	0.0149(6)	30.884(6)	21.108(4)
	Lin.	0.0160(8)	0.0161(8)	0.0140(6)	0.0141(6)	30.482(6)	20.634(4)
	Quad.	0.0189(8)	0.0190(8)	0.0166(6)	0.0167(6)	31.216(6)	21.491(4)
	Exp.	0.0313(8)	0.0315(8)	0.0275(6)	0.0276(6)	30.828(6)	21.040(4)
20°	Hom.	0.0336(11)	0.0342(11)	0.0277(8)	0.0280(8)	58.500(8)	38.725(5)
	Lin.	0.0317(11)	0.0323(11)	0.0262(8)	0.0265(8)	57.607(8)	38.049(5)
	Quad.	0.0378(10)	0.0385(11)	0.0311(8)	0.0314(8)	59.228(8)	39.275(5)
	Exp.	0.0624(11)	0.0634(11)	0.0513(8)	0.0519(8)	58.373(8)	38.629(5)
30°	Hom.	0.0477(12)	0.0491(12)	0.0369(9)	0.0376(9)	81.463(9)	53.729(6)
	Lin.	0.0451(12)	0.0464(12)	0.0349(9)	0.0356(9)	80.020(9)	52.646(6)
	Quad.	0.0536(12)	0.0551(12)	0.0415(9)	0.0422(9)	82.637(9)	54.326(5)
	Exp.	0.0884(12)	0.0910(12)	0.0685(9)	0.0697(9)	81.258(9)	53.575(6)
40°	Hom.	0.0577(13)	0.0600(13)	0.0420(10)	0.0430(10)	97.528(8)	61.953(6)
	Lin.	0.0543(13)	0.0564(14)	0.0396(10)	0.0405(10)	95.919(8)	60.864(6)
	Quad.	0.0650(13)	0.0676(13)	0.0471(9)	0.0484(10)	98.838(8)	62.839(6)
	Exp.	0.107(13)	0.111(13)	0.0778(10)	0.0797(10)	97.299(8)	61.798(6)
45°	Hom.	0.0605(14)	0.0631(14)	0.0423(10)	0.0435(10)	102.515(7)	63.871(5)
	Lin.	0.0568(14)	0.0592(14)	0.0398(10)	0.0410(10)	100.821(7)	63.105(6)
	Quad.	0.0684(14)	0.0713(14)	0.0476(10)	0.0490(10)	103.894(7)	64.435(5)
	Exp.	0.112(14)	0.117(14)	0.0783(10)	0.0807(10)	102.274(7)	63.773(5)
50°	Hom.	0.0615(14)	0.0645(14)	0.0414(10)	0.0429(10)	105.213(5)	63.741(5)
	Lin.	0.0577(14)	0.0605(14)	0.0390(10)	0.0404(10)	103.500(5)	62.959(5)
	Quad.	0.0695(14)	0.0729(14)	0.0467(10)	0.0483(10)	106.598(4)	64.381(5)
	Exp.	0.1138(14)	0.119(14)	0.0768(10)	0.0795(10)	104.969(5)	63.630(5)
60°	Hom.	0.0575(15)	0.0607(15)	0.0363(10)	0.0380(10)	106.0399(1)	59.127(4)
	Lin.	0.0538(15)	0.0567(15)	0.0341(10)	0.0356(10)	103.771(1)	58.386(4)
	Quad.	0.0651(15)	0.0687(15)	0.0410(10)	0.0429(10)	107.880(1)	59.645(3)
	Exp.	0.106(15)	0.112(15)	0.0673(10)	0.0703(10)	105.718(1)	59.021(4)
70°	Hom.	0.0448(15)	0.0475(16)	0.0270(10)	0.0285(10)	108.306(1)	48.882(1)
	Lin.	0.0418(15)	0.0444(16)	0.0253(10)	0.0266(10)	105.218(1)	48.155(1)
	Quad.	0.0508(16)	0.0539(16)	0.0306(10)	0.0322(10)	110.794(1)	49.476(1)
	Exp.	0.0829(15)	0.0880(16)	0.0501(10)	0.0527(10)	107.869(1)	48.779(1)

Table 6
Variations of the values of P^{TC}_{Hcr} (MPa), P^{TC}_{Lcr} (MPa), f_{TC} (Hz) and $n_{HTC}, n_{LTC}, n_{fTC}$ for H and NH orthotropic truncated conical shells with the ratio L/R_1 ($\gamma = 30°$, $R_1/h = 100$, $\mu = 0.9$).

$\varphi_j(\bar{\zeta})$	Hom.	$\bar{\zeta}$	$\bar{\zeta}^2$	$e^{-0.1\lvert\bar{\zeta}\rvert}\cos(0.5\bar{\zeta})$
L/R_1	P^{TC}_{Hcr} (MPa) and (n_{Hcr}); $\lambda = 4$; $j = 1$			
0.25	2.713(42)	2.530(42)	3.079(42)	5.022(42)
0.50	0.652(23)	0.608(23)	0.739(23)	1.207(23)
0.75	0.287(17)	0.269(17)	0.325(17)	0.533(17)
1.0	0.165(15)	0.155(15)	0.186(15)	0.306(15)
3.0	0.023(12)	0.022(12)	0.026(11)	0.044(12)
5.0	0.009(11)	0.008(11)	0.010(11)	0.017(11)
L/R_1	P^{TC}_{Lcr} (MPa) and (n_{Lcr}); $\lambda = 4$; $j = 1$			
0.25	2.887(44)	2.693(44)	3.277(44)	5.344(44)
0.50	0.692(24)	0.646(24)	0.785(24)	1.282(24)
0.75	0.304(18)	0.284(18)	0.344(18)	0.562(18)
1.0	0.173(15)	0.162(15)	0.195(15)	0.320(15)
3.0	0.024(12)	0.023(12)	0.027(12)	0.044(12)
5.0	0.009(11)	0.008(11)	0.010(11)	0.017(11)
L/R_1	f_{TC} (Hz) and (n_{fcr}); $\lambda = 2.4$; $j = 1, 2$			
0.25	1792.146(1)	1731.158(1)	1841.049(1)	1783.534(1)
0.50	473.674(1)	459.253(1)	485.275(1)	471.635(1)
0.75	248.149(1)	242.721(1)	252.549(1)	247.378(1)
1.0	177.391(5)	174.326(6)	179.713(5)	176.976(6)
3.0	50.678(9)	49.757(9)	51.428(9)	50.548(9)
5.0	27.324(9)	26.788(9)	27.720(8)	27.248(9)
L/R_1	f_{TC} (Hz) and (n_{fcr}); $\lambda = 2.4$; $j = 1$			
0.25	1792.146(1)	1731.158(1)	1908.841(1)	1122.309(1)
0.50	473.674(1)	459.253(1)	503.144(1)	451.368(1)
0.75	248.149(1)	242.721(1)	261.848(1)	293.855(1)
1.0	177.391(5)	174.326(6)	186.330(5)	215.822(6)
3.0	50.678(9)	49.757(9)	53.322(9)	64.507(9)
5.0	27.324(9)	26.788(9)	28.741(8)	35.614 (9)
L/R_1	f_{TC} (Hz) and (n_{fcr}); $\lambda = 2.4$; $j = 2$			
0.25	1792.146(1)	1792.146(1)	1728.499 (1)	1310.944(1)
0.50	473.674(1)	473.674(1)	456.852(1)	346.490(1)
0.75	248.149(1)	248.149(1)	239.336(1)	181.519(1)
1.0	177.391(5)	177.391(5)	171.091(5)	129.760(5)
3.0	50.678(9)	50.678 (9)	48.878(9)	37.071(9)
5.0	27.324(9)	27.324 (9)	26.353(9)	19.987(9)

of critical parameters are nearly same for parabolic and exponential cases, but percentage effects decrease for the linear case, i.e. for $\varphi_j(\bar{\zeta}) = \bar{\zeta}$; $(j = 1, 2)$. For example; as the ratio $L/R_1 = 0.25$ the percentage effects of the non-homogeneity on buckling pressures and lowest cyclic frequency are 6.75% and 3.40%, respectively, but as $L/R_1 = 5$ the percentage effects are 5.08% and 1.9%, respectively.

Fig. 5 shows variations of the values of hydrostatic buckling pressures for homogeneous and non-homogeneous orthotropic truncated conical shells with different non-homogeneity functions versus E_{01}/E_{02} ratio. The truncated conical shell has geometrical parameters as $L = 2$ m, $R_1 = 1$ m, $R_2 = 2$ m, $h = 0.01$ m, $\gamma = 30°$. It is seen that as E_{01}/E_{02} ratio is increased, the values of hydrostatic buckling pressures decrease for homogeneous and non-homogeneous orthotropic truncated conical shells with clamped edges. As the ratio E_{01}/E_{02} increases, the percentage effects on the hydrostatic buckling pressure for the homogeneous and non-homogeneous orthotropic truncated conical shells are nearly equal. When the variation of the Yong's moduli and the density are given by linear, quadratic and exponential functions, it is observed that the effect of this non-homogeneity on the hydrostatic buckling pressure is relatively more for the exponential function.

Fig. 6 shows variations of the values of fundamental cyclic frequencies for homogeneous and non-homogeneous orthotropic truncated conical shells in which Young's moduli and density vary together and separately, versus E_{01}/E_{02} ratio. The non-homogeneity function is parabolic. It is seen that as E_{01}/E_{02} ratio is increased, the values of fundamental cyclic frequencies decrease for homogeneous and non-homogeneous orthotropic truncated conical shells. As the ratio E_{01}/E_{02} increases, the percentage effects on the fundamental cyclic frequencies for homogeneous and non-homogeneous cases are nearly equal. When the Young's moduli vary together with the density in the thickness direction, the higher effect on the fundamental cyclic frequency is nearly 1.8%. When the only density varies in the thickness direction and the Young's moduli are kept constant the higher effect on the fundamental cyclic frequency is 3.55%. When the density is kept constant and only the Young's moduli are changed, the higher effect on the fundamental cyclic frequency is 5.2%.

The values of hydrostatic buckling pressures and fundamental cyclic frequencies for homogeneous and non-homogeneous orthotropic truncated conical shells are shown in Figs. 7 and 8, respectively, with respect to R_2/R_1, as the semi-vertex angle $\gamma = 30^\circ$ and $h = 0.01$ m. When the ratio R_2/R_1 is increased, the values of the hydrostatic buckling pressure and fundamental cyclic frequency decrease for homogeneous and non-homogeneous orthotropic truncated conical shells. When the non-homogeneous orthotropic truncated conical shells is compared with homogeneous orthotropic truncated conical shells; the percent changes in the values of the hydrostatic buckling pressure are 5.4%; 12.3% and 85% for the non-homogeneity cases as linear, parabolic and exponential, respectively. But the effect of the non-homogeneity on the values of the fundamental cyclic frequencies is not considerable. As the R_2/R_1 ratio increases, the percentage effects on the critical parameters for the homogeneous and non-homogeneous orthotropic truncated conical shells are nearly equal.

6. Conclusions

In this paper an analytical procedure is given to study the free vibration and stability characteristics of homogeneous and non-homogeneous orthotropic truncated and complete conical shells with clamped edges under uniform hydrostatic and lateral pressures. At first, the basic relations, the modified Donnell type stability and compatibility equations have been obtained for orthotropic truncated conical shells, the material properties of which vary continuously in the thickness direction. Applying Galerkin methods to the foregoing equations, the analytical formulas for buckling pressures and fundamental cyclic natural frequencies of non-homogeneous orthotropic conical shells are obtained. The appropriate formulas for homogeneous orthotropic and isotropic conical shells and for cylindrical shells made of homogeneous and non-homogeneous, orthotropic and isotropic materials are found as a special case. Finally, the influences of the non-homogeneity, orthotropy and the variations of conical shells characteristics on the critical lateral and hydrostatic pressures and natural frequencies are investigated, when Young's moduli and density vary together and separately. The results obtained for homogeneous cases are compared with their counterparts in the literature.

Appendix A

The expressions $\delta_{\bar{k}}, \Delta_{\bar{k}}, c_{ij},\ b_{ij}\ (\bar{k} = 1\text{—}16;\ i,j = 1\text{—}4)$ defined as follows:

$$\begin{gathered}
\delta_1 = c_{12};\quad \delta_2 = c_{11} - 4c_{12} - c_{22};\quad \delta_3 = 5c_{12} + 3c_{22} - 3c_{11} - c_{21};\quad \delta_4 = 2(c_{11} - c_{22} - c_{12} + c_{21});\\
\delta_5 = c_{21};\quad \delta_6 = c_{11} - 2c_{31} + c_{22};\quad \delta_7 = 4c_{31} - 3c_{11} - c_{22};\quad \delta_8 = 2(c_{11} - c_{31} + c_{21});\quad \delta_9 = c_{24};\\
\delta_{10} = c_{14} + c_{23} + 2c_{32};\quad \delta_{11} = 3c_{14} + c_{23} + 4c_{32};\quad \delta_{12} = 2(c_{14} + c_{32} + c_{24});\quad \delta_{13} = c_{13};\\
\delta_{14} = c_{23} - c_{14} + 4c_{13};\quad \delta_{15} = c_{24} - 3c_{23} + 3c_{14} - 5c_{13};\quad \delta_{16} = 2(c_{23} - c_{14} - c_{24} + c_{13});
\end{gathered} \tag{A1}$$

$$\begin{gathered}
\Delta_1 = b_{11};\quad \Delta_2 = 2b_{31} + b_{21} + b_{12};\quad \Delta_3 = 4b_{31} + 3b_{21} + b_{12};\quad \Delta_4 = 2(b_{31} + b_{21} + b_{11});\\
\Delta_5 = b_{22};\quad \Delta_6 = b_{21} - 4b_{22} - b_{12};\quad \Delta_7 = 5b_{22} + 3b_{12} - b_{11} - 3b_{21};\\
\Delta_8 = 2b_{21} - 2b_{22} - 2b_{12} + 2b_{11};\quad \Delta_9 = b_{14};\quad \Delta_{10} = 2b_{32} - b_{13} - b_{24};\\
\Delta_{11} = b_{13} + 3b_{24} - 4b_{32};\quad \Delta_{12} = 2b_{32} - 2b_{24} - 2b_{14};\quad \Delta_{13} = b_{23};\quad \Delta_{14} = b_{13} - b_{24} + 4b_{23};\\
\Delta_{15} = b_{14} - 3b_{13} + 3b_{24} - 5b_{23};\quad \Delta_{16} = 2b_{13} - 2b_{24} + 2b_{23} - 2b_{14}
\end{gathered} \tag{A2}$$

$$\begin{gathered}
c_{11} = a_{11}^1 b_{11} + a_{12}^1 b_{21};\quad c_{12} = a_{11}^1 b_{12} + a_{12}^1 b_{22};\quad c_{13} = a_{11}^1 b_{13} + a_{12}^1 b_{23} + a_{11}^2;\\
c_{14} = a_{11}^1 b_{14} + a_{12}^1 b_{24} + a_{12}^2;\quad c_{21} = a_{21}^1 b_{11} + a_{22}^1 b_{21};\quad c_{22} = a_{21}^1 b_{12} + a_{22}^1 b_{22};\\
c_{23} = a_{21}^1 b_{13} + a_{22}^1 b_{14} + a_{21}^2;\quad c_{24} = a_{21}^1 b_{14} + a_{22}^1 b_{13} + a_{22}^2;\quad c_{31} = a_{66}^1 b_{31};\\
c_{32} = a_{66}^1 b_{32} + a_{66}^2;\quad b_{11} = a_{22}^0/L_0;\quad b_{12} = -a_{12}^0/L_0;\quad b_{13} = \left(a_{12}^0 a_{21}^1 - a_{11}^1 a_{22}^0\right)/L_0;\\
b_{14} = \left(a_{12}^0 a_{22}^1 - a_{12}^1 a_{22}^0\right)/L_0;\quad b_{21} = -a_{21}^0/L_0;\quad b_{22} = a_{11}^0/L_0;\quad b_{23} = \left(a_{21}^0 a_{11}^1 - a_{21}^1 a_{11}^0\right)/L_0;\\
b_{24} = \left(a_{21}^0 a_{12}^1 - a_{22}^1 a_{11}^0\right)/L_0;\quad b_{31} = 1/a_{66}^0;\quad b_{32} = -a_{66}^1/a_{66}^0;\quad L_0 = a_{11}^0 a_{22}^0 - a_{12}^0 a_{21}^0.
\end{gathered} \tag{A3}$$

in which expressions $a_{ij}^k,\ k = 0,1,2;\ i,j = 1,2,6$ are defined as follows:

$$\begin{gathered}
a_{11}^k = \frac{E_{01}h^{k+1}}{1 - \nu_{12}\nu_{21}} \int_{-1/2}^{1/2} \bar{\zeta}^k \left[1 + \mu\varphi_1(\bar{\zeta})\right] d\bar{\zeta},\quad a_{22}^k = \frac{E_{02}h^{k+1}}{1 - \nu_{12}\nu_{21}} \int_{-1/2}^{1/2} \bar{\zeta}^k \left[1 + \mu\varphi_1(\bar{\zeta})\right] d\bar{\zeta},\\
a_{12}^k = \nu_{21} a_{11}^k = a_{21}^k = \nu_{12} a_{22}^k,\quad a_{66}^k = 2G_0 h^{k+1} \int_{-1/2}^{1/2} \bar{\zeta}^k \left[1 + \mu\varphi_1(\bar{\zeta})\right] d\bar{\zeta},\quad k = 0,1,2,\\
\rho_1 = \rho_0 \int_{-1/2}^{1/2} \left[1 + \mu\varphi_2(\bar{\zeta})\right] d\bar{\zeta}.
\end{gathered} \tag{A4}$$

Appendix B

In Ref. [37] the natural frequency $\bar{\omega}_{TC}$ for the elliptical truncated conical shell is defined as

$$\bar{\omega}_{\mathrm{TC}} = \Omega_2 \sqrt{\frac{D_0}{\rho_0 h R_2^4}} \sqrt{\frac{4m_1^2 + (\lambda+1)^2}{4m_1^2 + (\lambda-1)^2}} \sqrt{\frac{1 - e^{-2(\lambda-1)x_0}}{1 - e^{-2(\lambda+1)x_0}} \frac{m_1^2 + (\lambda+1)^2}{m_1^2 + (\lambda-1)^2} \frac{\lambda+1}{\lambda-1}}, \tag{B1}$$

where the following definitions apply:

$$\Omega_2^2 = \frac{16}{3} m_1^4 + \frac{8}{3} m_1^2 n_1^2 \frac{\varepsilon_2}{\varepsilon_3} + n_1^4 \frac{\varepsilon_1}{\varepsilon_3} + \left[\frac{2}{3}(\lambda^2 - 2\lambda - 2)\frac{\varepsilon_2}{\varepsilon_3} - \frac{1}{\sin^2\gamma_0}\frac{\varepsilon_1'}{\varepsilon_3}\right] n_1^2 + \frac{8}{3} m_1^2 (2\lambda^2 - 4\lambda + 3) + \lambda^2 (\lambda-2)^2$$
$$+ \frac{8}{3}(\lambda-1)^2 + \frac{\eta_2^2 \left(\frac{1-\varepsilon^2}{3}\frac{\varepsilon_4}{\varepsilon_3}\right)^2 (4m_1^2 + \lambda^2)^2}{\frac{16}{3} m_1^4 + \frac{8}{3}\frac{\varepsilon_2}{\varepsilon_3} m_1^2 n_1^2 + n_1^4 \frac{\varepsilon_1}{\varepsilon_3} + \varepsilon_7 n_1^2 + \frac{8}{3} m_1^2 (2\lambda^2 + 1) + \lambda^4 + \frac{2}{3}\lambda^2 + 1}, \tag{B2}$$

$$\varepsilon_7 = \frac{2\varepsilon_2(\lambda^2 - 2\lambda - 2)}{3\varepsilon_3} - \frac{1}{\sin^2\gamma_0}\frac{\varepsilon_1}{\varepsilon_3}; \quad \eta_2^2 = \eta_1^2 \frac{4m_1^2 + (\lambda-1)^2}{4m_1^2 + \lambda^2};$$

$$\eta_1^2 = \eta^2 \frac{1 - e^{-2\lambda x_0}}{1 - e^{-2(\lambda-1)x_0}} \frac{m_1^2 + (\lambda-1)^2}{m_1^2 + \lambda^2} \frac{\lambda-1}{\lambda}; \quad \eta^2 = \frac{E_0 h S_2^2}{D_0 \tan^2\gamma}; \quad D_0 = \frac{E_0 h^3}{12(1 - \nu_0^2)};$$

The elliptical truncated conical shell is transformed into the circular truncated conical shell when $\varepsilon = 0$. If $\varepsilon = 0$ is substituted in Eq. (B1) corresponding formula for the natural frequency of the circular truncated conical shell are obtained. In this case the following definitions apply:

$$\varepsilon_1' = 0; \quad \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = \varepsilon_6 = 1; \quad m_1 = \beta_1; \quad n_1 = \beta_2; \quad \gamma_0 = \gamma. \tag{B3}$$

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