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Investigation of 9th Grade Students' Geometrical Figure Apprehension

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Abstract: In the present study, the aim was to investigate 9th grade students' geometrical figure apprehension. To this end, the Figure Apprehension Cognitive Processes Test (FACPT), constructed by the researchers of the study, was administered to 51 ninth grade students, with whom clinical interviews were also conducted. As a result of the data analysed, it was found that the perceptual, discursive and operative types of apprehension of more than half of the students were not at enough level for high school geometry. Most of the students were found to be unsuccessful in recognizing the various sub-figures present within a geometric figure, in transforming verbal information to visual information, in deriving at verbal information based on visual information, in arriving at conclusions without being influenced by the appearance of a figure, and in decomposing and recomposing geometric figures. This shows that teachers need to focus on not only conceptual knowledge but also the structure of the figure apprehension processes of students prior to geometry classes.

Keywords: *Geometric reasoning, duval's cognitive model, geometric figure apprehension.*

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Introduction

The Geometric figures present a holistic representation of mathematical relationships within a problem. The identification of geometric relationships based on these representations is the most critical step in the problem-solving process (Parzyz, 1991). Since geometric figures, by their very nature, cannot be considered independent of concepts (Fischbein, 1993), students need to establish accurate relationships between a given geometric figure and mathematical principles (concepts) while solving problems or providing proof. According to Duval, such an interaction can be realized by undergoing the experience of figure apprehension processes. Each of these processes, named as perceptual, discursive, operative and sequential perceptions, serves a purpose within the figure-concept interaction (Duval, 1995, 1998). Thus, some studies reveal the importance of students' figure apprehension processes while solving problems or providing proof (Llinares & Clemente, 2014; Torregrosa & Quesada, 2008).

At the end of secondary school in Turkey, students are expected to be able to explain the features of geometric properties and the relationships among them, to provide empirical evidence and to use this information in constructing, analysing and categorizing geometric figures (MoNE, 2013b, 2018b). In high school, students are expected to be able to prove the geometric relationships they had been experimentally proving in primary school or understand the proofs made (MoNE, 2013a, 2018a). Such an expectation means that 9th grade students' geometrical figure apprehension needs to be at a certain level. Otherwise, students will not be able to establish accurate interactions between figures and mathematical principles, and hence, will based their inferences on the appearance of figures (Duval, 1995, 1998). However, when the literature is reviewed, it can be seen that there are very few studies that have analysed high school students' figure apprehension processes (Michael, 2013; Michael, Gagathis, Avgerinos & Kuzniak, 2011). In the few studies in this area, comparative analyses of figure apprehension processes were conducted (Michael, 2013) with specific focus on the operative apprehension (Michael, Gagathis, Avgerinos & Kuzniak, 2011) of students in different grades (grades 9, 10 and 11). The results obtained include important information regarding how students' figure apprehension varies from one grade level to another. However, it is believed that new studies on samples from different countries will also be of significant contribution to the literature. Thus, the present study will be of significance and it will also serve to reveal the readiness level of newly-enrolled high school students in the geometry lesson. The structure of high school students' figure apprehension processes can be better understood and the obtained

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results can guide teachers during their formation of class activities. For these reasons, the present study aims to investigate 9th grade students' figure apprehension processes.

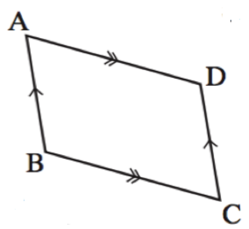
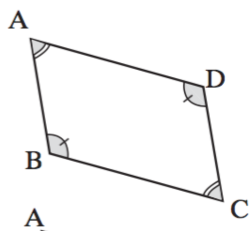
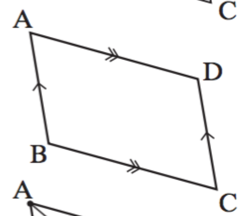
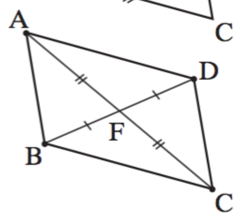
Theoretical Framework

Different approaches have emerged regarding efforts to equip individuals with higher order thinking skills that geometry can provide, the nature of geometry, and how it can be taught. One of these approaches is Duval's Cognitive Model. In this model, geometry is addressed from perceptual and cognitive viewpoints and certain processes have been put forward (Jones,1998). Duval tried to explain which processes are experienced during figure apprehension and named these processes as figure apprehension processes. In Duval's framework, figure apprehension processes are comprised of four types of apprehension: perceptual, discursive, sequential and operative apprehension. According to Duval, each of these processes serves various functions to enable the mathematical relationships in a figure to be recognized (observed) and functions interactively during a problem-solving process. To accurately realize this interaction, figure apprehension processes should be developed independent of each other in unique ways (Duval,1995).

Perceptual apprehension is the stage at which information can be obtained at first glance of a figure, and information regarding a geometric shape's figural structure (configuration). It includes such processes as providing information about the name and area of a figure, and recognizing the fundamental geometric elements of a figure (point, line segment, triangle, circle etc.). Identification of a figure's sub-elements is also among the perceptual apprehension process. This type of apprehension is static, and the relationships between this apprehension and the sub-figures cannot be recognized (Duval, 1995).

It impossible to identify the mathematical properties of a geometric figure based solely on perceptual apprehension. Some preliminary information about the figure is also needed. Establishing a relationship between the figure and mathematical principles (definition, theorem, axiom etc.) based on the given preliminary information is named as discursive apprehension (Duval, 1995, 1998; Micheal, 2013). The interaction between discursive apprehension and perceptual apprehension is as follows (Duval, 1998):

Table 1. The Relationship between Discursive Apprehension and Perceptual Apprehension

Transition from Perceptual Apprehension to Discursive Apprehension	Transition from Discursive Apprehension to Perceptual Apprehension
 "ABCD is a parallelogram." The verbal information above, i.e. ABCD being presented as a parallelogram, leads to the dominance of verbal information over the figure. In this way, the sides (e.g. opposite sides are parallel), corners, and angles gain more focus, and the relationships among them can be observed.	"Let ABCD be a parallelogram…" With this verbal information, numerous different types of perceptual apprehension that meet the requirements for a parallelogram can be formed. For example;   

Unlike the drawings made by hand, constructing geometric figures by means of a tool, that is creating a model of a figure, will cater to acquiring information about the figure and this information will serve to recognize the geometric relationships in a problem. For this reason, in Duval's Cognitive Model, constructing a geometrical figure by means of a tool or defining the processes of constructing a figure can be observed among the figure apprehension processes; these processes are named as sequential apprehension. On the other hand, the changes made on the initial appearance of a figure (drawing supplementary lines, decomposing the figure into its components and recomposing them into some other figure, changing the figure's position and direction etc.) and the act of focusing on some components of the figure more than on others are referred to as operative apprehension (Duval, 1995).

When all of this is taken into consideration, the indicators of figure apprehension processes with respect to student behavior can be presented as follows:

Table 2. The Indicators of Figure Apprehension Processes

Perceptual Apprehension	Discursive Apprehension	Operative Apprehension	Sequential Apprehension
Can recognize the given figure and its basic geometrical elements and can name them.	Can accurately transform the given verbal information (the information given about the figure in the question, symbolic representation and concepts) into visual information. Does not get misguided by the appearance of the figure while making inferences about geometric relationships. Can accurately transform the visual information given on a figure into verbal information by using symbols, notations and mathematical concepts and can make correct inferences.	Can decompose a given geometric figure and recompose the components to form a different figure. Can focus on some parts of a figure and can change the figure by adding or omitting new geometrical elements. Can change the position or direction of a given figure or its sub-elements.	Can construct a geometric figure by means of a tool. Can define how to construct a geometrical figure by means of a tool.

When the indicators of figure apprehension processes are examined, it can be observed that geometric thinking is addressed as cognitive and apprehension processes rather than as the teaching of topics and concepts. The reason is that mathematics is not a branch of science that is nothing more than mathematical results and concepts found by somebody; it is a way of thinking (Cuoco, Goldenberg & Mark, 1996). This way of thinking is established by means of cognitive and apprehension processes an individual engages in (Duval, 1998).

Methodology

The present research is a qualitative study aiming to identify 9th grade students' figure apprehension processes.

Study Group

The study group was comprised of 51 ninth grade students. The study was implemented in a state high school (Anatolian High School) in Turkey which selects its students based on a centralized national exam and is a moderate-level school in its location in terms of student scores. All the 9th grade students in this high school were included in the intervention. Anatolian High Schools aim to equip students with academic skills and are, except for vocational schools, the most preferred type of high school among all the other types of high schools. Such a school was selected for the study so that a relative majority of the students in Turkey could be represented. In secondary school, these students learned such geometry topics as triangles, polygons, geometric objects and transformation geometry within the geometry learning sub-domain. The study was implemented before the 9th grade geometry subjects were taught to the students. In this way, the study attempted to reveal the structure of the students' figure apprehension processes before starting their geometry courses in high school.

Data Collection

The data of the current study were obtained by means of the Figure Apprehension Cognitive Processes Test (FACPT), developed by the researcher. The Test was developed by identifying the geometric figure apprehension indicators (see Table 2) based on Duval's Cognitive Model. Taking these indicators into consideration, five open-ended questions, one of which (the question used to identify students' operative apprehension) was taken from a study by Michael (2013) (Question 5 in Appendix), were prepared by receiving the opinions of two experts, both of whom held a Ph.D. degree.

During the preparation of the test, special care was given ensure that the problem statements were clear, suitable to the students' level (by considering the curriculum used in Turkey) and have the potential to reveal the identified indicators. The prepared questions were piloted to observe whether they served the purpose of the research study in same school the previous year. In the pilot study, one question was left unanswered by most of the students. After examining the questions, it was decided that the question was not compatible to the level of the students. Hence, the question was replaced with another one and during this process, expert opinion was received to finalize the test (see Appendix 1). The content of the questions and the related indicators are presented in Table 3 as follows:

Table 3. The content of the questions and the indicators they sought

	The content of the questions	**Indicators sought**	**Figure apprehension processes**
Q1	In this question, students are given a geometric figure and asked to write what sub geometric figures it is comprised of. Thus, the aim was to measure whether students recognized the basic geometric figures that made up the given figure.	Can recognize and name the given figure and its basic geometric elements.	Perceptual apprehension
Q2	In this question, the students were asked to write the mathematical properties that could definitely be inferred from the geometric figure given together with the symbols indicating certain mathematical properties. The aim, thus, was to determine whether or not the students could accurately transform the given visual information to verbal information and use this verbal information to make accurate inferences.	Does not base inferences regarding geometric relationships on deceptive interpretations of the figure's appearance. Can accurately transform the visual information given on the figure to verbal information by using the notations and mathematical concepts, and can make accurate inferences.	Transition from Perceptual Apprehension to Discursive Apprehension
Q3	In this question, the students are given a figure and some verbal information related to the figure, and are asked to present the verbal information on the figure by using correct notations. The aim, thus, was to determine whether or not the students could accurately transform verbal information to visual information.	Can accurately transform the given verbal information to visual information (figure-related information in the question, symbolic notations and concepts).	Transition from Perceptual Apprehension to Discursive Apprehension
Q4	In this question, the students were asked to use the given tools (such as plotting paper, ruler, compass) to construct the given geometric figure and define the construction process. The aim, thus, was to measure whether or not the students could construct a figure with the aid of a tool and define the construction process.	Can construct a geometrical figure by means of a tool. Can define the construction of a geometrical figure by means of a tool.	Sequential apprehension
Q5	In this question, the students are given two geometric figures and asked to compare the areas of these figures. To do this, they are expected to change the initial appearances of the figures to arrive at an outcome. The aim, thus, was to measure if students could make changes on the given figures.	Can focus on some components of a figure and can change the figure by adding or omitting new geometric elements. Does not need quantitative data to make changes on the given geometric figure (for example, self-assigning a numerical value for a missing side length). Can change the position or direction of a given figure or its sub-components. Can decompose a given geometrical figure and can recompose the components to construct a new figure.	Operative apprehension

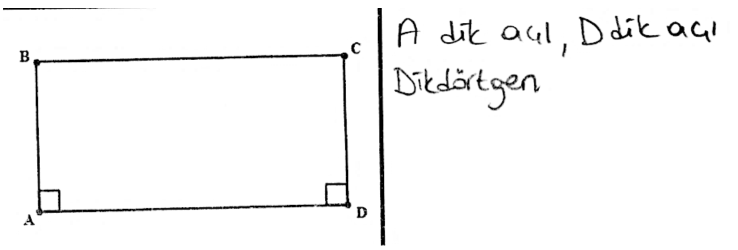
The prepared questions were administered to the students in one class hour during the second term of the 2014-2015 academic year before the geometry topics were taught. After the written responses were collected from the students, Pre-interviews was conducted with some volunteer students and 12 students who could express their opinions explicitly were chosen from different categories to hold approximately 20-to-25-minute clinical interviews. Clinical interviews were conducted because by means of these interviews, data on students' cognitive processes regarding their opinions and understanding could be collected and analysed, and the rationale underlying their opinions could be revealed (Clements, 2003). The interviews were recorded, with the permission of the student, via a voice recorder. In these interviews, the students were asked to explain and justify their responses via the question, "Could you explain and justify the response you have given?" Thus, the interviewer tried to reveal the justifications underlying the written responses.

Analyzing of Data

To be able to evaluate the responses given to the test (FACPT), the categorical scoring rubric was constructed by receiving the expert opinions of two mathematics educators (see Appendix 1). The categorical scoring rubric was constructed by making use of the indicators of the figure apprehension processes obtained from the pilot study. By using the indicators, the category to include the highest score was identified. Subsequently, by taking this category as a reference, the student responses in the pilot study were analysed to establish the other lower scores. In this scoring system, the highest score represents the highest level of performance expected of the student. Thus, the lower the score is, the lower is the quality of the student's figure apprehension processes. The student responses were scored based on the pre-prepared categorical scoring rubric by the researchers independent of each other. When there was a discrepancy in the scoring of the student responses, those responses were readdressed to arrive at a common decision in the scoring.

The percentage of students for each category was identified, and based on these percentages, the researchers tried to reveal students' figure apprehension processes. The data analysis process for each student response is as follows:

Table 4. A student's written response to the second question of the Figure Apprehension Cognitive Processes Test and the categorical scores

Student Responses	Categorical Scores
 Figure 1. Student's response (A is right angle, D is right angle Rectangle)	0: Unanswered or Irrelevant responses (does not transform the given visual information to verbal information and makes inaccurate inferences). 1: Transforms visual information to verbal information accurately, but makes inaccurate inferences. 2: Transforms visual information to verbal information accurately, but does not make any inferences. 3: Does not transform visual information to verbal information, but makes accurate inferences. 4: Transforms visual information to verbal information accurately, and makes accurate inferences.

When the student response in Figure 1 is examined, it can be observed that regarding the figure, the student drew such conclusions about the figure as "A is a right angle," "D is a right angle," and "rectangle". This shows that the student transformed the visual information given on the figure to verbal information accurately but drew a conclusion that cannot be directly drawn from the given verbal information. Based on the categories defined for the question, it can be observed that this response can be explained with the indicator in the 1-point category: "Can accurately transform visual information to verbal information but makes inaccurate inferences." Hence, 1 point was given to the student for this question. The written responses for the other questions were scored in a similar way.

Findings

As a result of the analyses of the data obtained from the FACPT by means of the categorical scoring rubric in the current study, in which the geometrical figure apprehension processes of 51 ninth grade students were investigated, it was revealed that the majority of students could not display the behaviour expected of the highest category in all the figure apprehension processes, except the sequential apprehension. The findings regarding each type of apprehension are presented below in detail under some sub-headings.

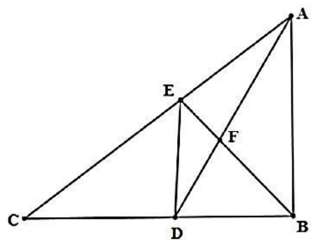
Students' Perceptual Apprehension

To identify the structure of students' perceptual apprehension, the students were given a figure and asked to identify the geometric shapes within that figure. The responses given were analysed based on three categories, and the percentage of students in each the category was identified. The obtained results are as follows:

Table 5. Students' Perceptual Apprehension Distribution across the Categorical Scores

Question

What geometric shapes form the given figure? Write them by considering the corners.



Indicator	Categorical Scoring Rubric	
Can recognize and name the given geometric figure and the basic geometric elements making up the figure.	0: No figure is mentioned. 1: Only the triangles are mentioned. 2: Different geometric figures are written (triangle, quadrangle, pentagon…)	
Categorical Scores	**Number of Students**	**Percentage**
0	1	1.96%
1	31	60.78%
2	19	37.25%
	Total:51	Total: 100%

As can be observed in the table, a majority of the students (60.78%) are within category 1. On the other hand, only about 37% of the students could be placed in the highest category. This showed that the majority of the students had written triangle as a response and, thus, had limited perceptual apprehension. During the interviews, when these students were asked why they had written only 'triangle', some of them responded by saying that they could not think of writing a quadrangle, while others stated that they could not see a quadrangle in the figure. An excerpt from an interview with a student in category 1 is presented below:

Researcher: *Which geometric figures can you see in the given figure?*

Student: *The CED, CAB, EFD, AFB, DFB, EFA triangles.*

Researcher: *You've written only triangles. Why didn't you write quadrangles? Couldn't you recognize them on the figure?*

Student: *Yes, now that you've mentioned it, I recognize it. It never occurred to me to write a quadrangle.*

As can be understood from the given interview excerpt, the student could only write the triangles presented on the figure. However, when the student was reminded of other geometric figures, s/he could see that other figures, apart from the triangle, could have been written. In another interview with a student from the 1-point category, the student reported that s/he could not see a quadrangle or any other figure even though s/he was reminded of other geometric figures. This demonstrates that the perceptual apprehension of the majority of the students is limited to triangles.

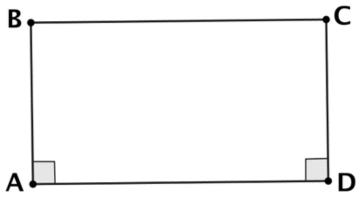
Transition from Perceptual Apprehension to Discursive Apprehension

To identify students' transition from perceptual to discursive apprehension, a figure resembling a rectangle with two right angles was used. The students were asked to write the mathematical properties that were relevant to the given figure. The given responses were analysed under four categories. The results obtained are as follows:

Table 6. The Categorical Score Distribution of Students' Transition from Perceptual Apprehension to Discursive Apprehension

Question

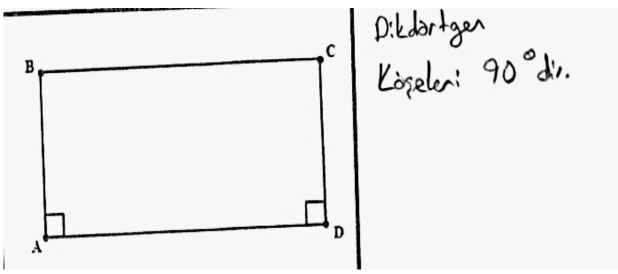
Write the mathematical properties of the figure by considering the given figure and the notations given on it. (*You can write more than one feature.)*



Indicator	**Categorical Scoring Rubric**
The ability to transform visual information to verbal information accurately and make correct inferences	0: Other responses: Makes inaccurate inferences based on the appearance of the figure or leaves the question unanswered. 1: Transforms visual information to verbal information accurately but makes incorrect inferences. 2: Transforms visual information to verbal information accurately but does not make any inferences. 3: Does not transform visual information to verbal information but makes correct inferences. 4: Transforms visual information to verbal information accurately and makes correct inferences.

Categorical Scores	**Number of Students**	**Percentage**
0	19	37.25%
1	12	23.53%
2	20	39.22%
3	0	0.00%
4	0	0.00%
	Total=51	Total=100%

When the results were examined, it was observed that none of the students were in the higher categories of 3 or 4. All the students were in one of the lower categories, namely 0 (zero), 1 or 2. This indicates that the students are unsuccessful in both transforming visual information to verbal information and using this information to make inferences. For example, when the written responses of the students in category of 0 (zero) point were examined, it could be revealed that the students could not produce verbal information based on the signs given on the figure and that they made incorrect inferences. The written response of one of these students is as follows:



(Rectangle, the corners are 90 degrees)

Figure 2. The written response of a student in the zero-point category

In the response given, the student has directly written the mathematical properties of the figure. This demonstrates that the there is no transition from perceptual apprehension to discursive apprehension; that is, it indicates that the mathematical properties expressed visually through the figure were not identified. In the interviews, it was observed that the students did not make use of their discursive apprehension, but based their inferences solely on perceptual apprehension. Thus, they wrote the properties of the figure as the mathematical properties to support their inferences. An excerpt from an interview with one of these students is as follows:

Researcher: *Could you talk about the mathematical properties of the figure below by looking at the given figure and the marks given on the figure?*

Student: *Figure is rectangle and the corners are 90 degrees.*

Researcher: *Why do you think it's a rectangle?*

Student: *The opposite sides are equal and the angles are 90 degrees.*

Researcher: *Information is not given only about some angles; for instance, angles C and B are not 90 degrees. Why did you think that the other angles were 90 degrees too?*

Student: *Because in regular quadrangles they are always 90 degrees.*

Researcher: *What should regular quadrangles mean to us?*

Student: *For example, square, rectangle*

It can be observed in the interview that the student directly arrived at the conclusion that the figure was a rectangle and when asked to justify his/her response, s/he listed the properties of a rectangle. As can be understood, here the student's perceptual apprehension, that is, the appearance of the figure, constitutes the primary source which the conclusion drawn is based on. However, the student should have transformed the visual data of the figure to verbal information expressing the mathematical properties (such as angles A and D are 90 degrees, sides AB and DC are parallel) and should have written the essential conclusions that could be drawn from this information.

When the written responses of the students in the 1-point category were examined, it was revealed that these students accurately transformed visual information to verbal information but still made incorrect inferences. Thus, this indicates that verbal information is not sufficient to make accurate inferences. In the interviews conducted with these students, it was observed that, just like the students in the 0 (zero)-point category, they made inferences based on the appearance of the figure. In other words, students' perceptual apprehension dominated the inference making process. When the written responses provided by the students in the 2-point category were examined, it was revealed that these students had accurately transformed the information on the figure to verbal information but had not made any inferences. Unlike the students in the 1-point category, these students had not written any mathematical properties regarding the figure, apart from the verbal information they had obtained. However, it was observed in the interviews that similar to the students in the 0- and 1-point categories, these students were under the influence of their perceptual apprehension while making inferences. Thus, when these students were asked questions about the figure during the interview, it was understood that these students were driven by their perceptual apprehension. Below is a written response and an excerpt of the interview with a student from the 2-point category:

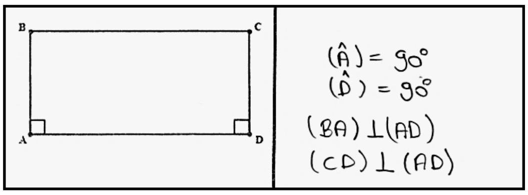


Figure 3. The written response of a student in the 2-point category

Researcher: *You've written that angles A and D are 90 degrees? Don't you think there are other mathematical properties in the figure? For example, what can you say about angles B and C?*

Student: *Well, I think angles B and C are also 90 degrees.*

Researcher: *Do you think all the angles are 90 degrees?*

Student: *Yes. It forms a rectangle.*

Researcher: *How did you understand the figure is a rectangle?*

Student: *It can be seen.*

In the interview excerpt, the student thinks that not only angles A and D, but angles B and C are also 90 degrees. Furthermore, the student, who states that the figure is a rectangle, bases his/her justification on the appearance of the figure. This indicates that some of the students from the 2-point category are also under the influence of their perceptual apprehension while making inferences.

Transition from Discursive Apprehension to Perceptual Apprehension

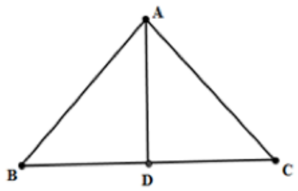
To be able to identify the students' transition from verbal apprehension to perceptual apprehension, the students were given the mathematical properties of a geometric figure verbally and asked to display these on the figure by using visual signs. The findings obtained from the student responses which were evaluated by using four categories are as follows:

Table 7. The Categorical Score Distribution of Students' Verbal Apprehension to Perceptual Apprehension

QUESTION

Try to display the information given below on the geometric figure by making use of notations.

- $|AB| = |AC|$
- $|AC| = |BC|$
- $[AD] \perp [BC]$
- $m(\hat{C}) = m(\hat{B})$
- $[AD]$, is the angle bisector of angle BAC.
- $[AD]$ is a median.



Indicator	Categorical Scoring Rubric
Can accurately transform the given verbal information (information related to the figure in the question, symbolic notations and concepts) to perceptual information.	0: The question is left unanswered. 1: Uses correct notations for some verbal information, but incorrect ones for others. 2: Uses correct notations but does not display some of the information on the figure. 3: Uses the correct notations and displays the given information on the figure accurately.

Categorical Scores	Number of Students	Percentage
0	1	1.96%
1	20	39.22%
2	28	54.90%
3	2	3.92%
	Total=51	Total=100%

Only approximately 4% of the students could transform the verbal information into visual information by using correct notations. The majority of the students (approximately 95%) had used incorrect notations or were unable to transform some of the verbal information into visual information. The written responses given by some of the (1-point category) students who used inaccurate notations are as follows:

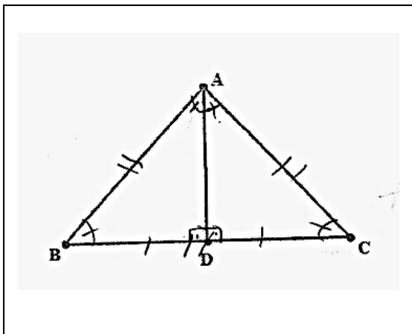


Figure 4a

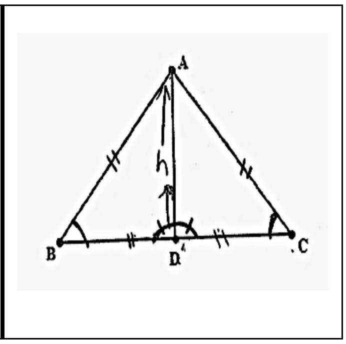


Figure 4b

When the written responses given in Figures 4a and 4b are examined, it can be observed that in Figure 4a, the student transformed the verbal information into visual information accurately with respect to the lengths of the sides in the triangle (e.g. the length of line segment AB is equal to that of line segment AC) but an inaccurate transformation was made with respect to the angles B, BAD, DAC and C. On the other hand, in Figure 4b, the student transformed verbal information into visual information correctly regarding angles B and C being equal but made incorrect transformations in terms of the lengths of the line segments BA, AC, BD and DC. The newly formed condition in this way started to include visual information that caused the person viewing the figure to arrive at inaccurate mathematical results. Thus, in the interviews conducted with students in the 1-point category, it was observed that when the students were given their self-constructed visual information, they arrived at inaccurate mathematical results. An excerpt from an interview conducted with one of these students is as follows:

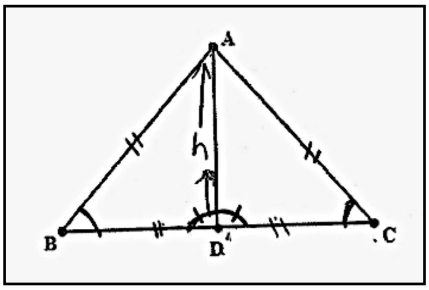


Figure 5. The figure formed by the student by transforming the given verbal information to perceptual information

Researcher: *Can you write the mathematical properties of the given figure (The figure constructed by the student, see Figure 5) by considering the notations on it?*

Student: *Here sides AC and DC are given as equal lengths...*

Researcher: *What else can be said?*

Student: *[The length of] Sides BD, DC and AB are equal. Also, there are two equal angles over there at [point] D (shows angles ADB and ADC).*

The student arrived at inaccurate results regarding the figure as s/he formed the same visual information for different verbal information. This shows that the student is unsuccessful even in a basic behavior as such required while solving a problem in geometry. When Table 7 is examined, it can be observed that 55% of the students in the 2-point category did not transform some of the verbal information to visual information. This demonstrates that more than half of the students did not use some of the given verbal information. During the interviews when students were asked why they had not transformed some of the verbal information into visual information, they stated that they did not read every piece of verbal information. An excerpt from an interview is as follows:

Researcher: *You haven't shown on the figure that side AD is a median.*

Student: *Oh, I didn't read that part, I didn't see it.*

Researcher: *Why didn't you read it?*

Student: *I don't read all the information while I'm solving a problem; it doesn't cause a problem. The information is already given on the figure.*

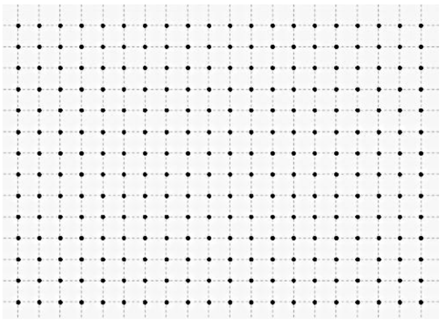
The student stated that, based on the habit acquired while solving geometry problems, s/he did not read some of the given information and thus could not show some of the information on the figure.

Students' Sequential Apprehension

To identify students' sequential apprehension, they were asked to draw a parallelogram and one of its heights by using the given isometric paper. The student responses evaluated based on two categories are as follows:

Table 8. The Categorical Score Distribution of Students' Sequential Apprehension

QUESTION
On the given isometric paper, draw a parallelogram and one of its heights.



Indicator	**Categorical Scoring Rubric**
Constructing a Geometric Figure with the Aid of a Tool and Explaining the Construction Process	0: Leaves the question unanswered or cannot draw a parallelogram and one of its heights. 1: Accurately draws a parallelogram and one of its heights and correctly explains how the drawing is made.

Categorical Scores	**Number of Students**	**Percentage**
0	7	7.69%
1	44	92.31%
	Total=51	Total=100%

In the table, a majority of the students (approximately 93%) did not experience difficulty in drawing a parallelogram by using unit squares. Considering that sequential apprehension refers to the relationship between the device used to construct a figure and the mathematical properties of the figure, it can be understood that students can establish accurate relationships between unit squares and the mathematical principles of a parallelogram. This finding was also confirmed in the interviews conducted with the students. When the students were asked how they had constructed the figure, their responses indicated that they could establish a relationship between the properties of a parallelogram and unit squares. A written response of a student and an excerpt from the interview conducted with the same student are as follows:

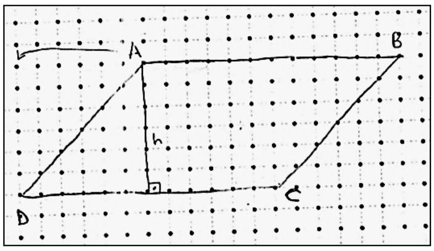


Figure 6. A student's written response from one of the score categories

Researcher: *How did you draw the parallelogram?*

Student: *I made sure the parallel sides were of equal length. For example, sides AB and DC are equal [in length], and sides AD and BC are equal [in length].*

Researcher: *How did you make sure that the sides were parallel?*

Student: *Well, wouldn't it be parallel if I drew them like this? (Shows his/her own drawing.)*

Researcher: *I don't know. What do you think?*

Student: *Well, if I take these lengths (shows the upper and lower bases) so that they don't intersect each other, they will be parallel.*

Researcher: *What did you do to draw the height?*

Student: *While drawing the height, I followed the points; I tried to draw a perpendicular line to DC.*

In the interview excerpt, the student was aware that the opposite sides of a parallelogram were equal and parallel to each other, and thus, with the aid of the unit squares, tried to construct these properties. However, since the students in the 0-point category could not establish this relationship, they could not draw the parallelogram nor one of its height accurately.

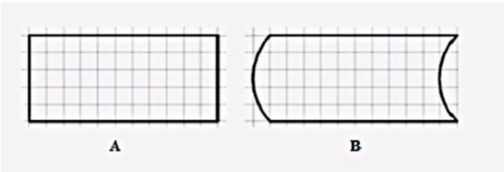
Students' Operative Apprehension

To identify students' operative apprehension, the students were given two figures and asked to compare their areas. To do this, they were expected to change the initial appearance of the figure. The written outcomes obtained were evaluated under two categories, and the following findings were obtained:

Table 9. The Categorical Score Distribution of Students' Operative Apprehension

QUESTION

Which of the following is true regarding the areas of figures A and B?
The area of A is greater than area of B
The area of A is equal area of B
The area of A is smaller than B
Can't comment because no length is given

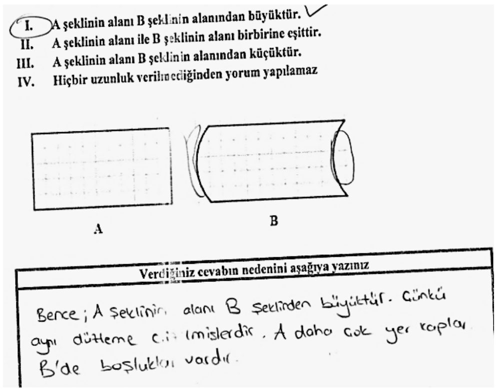


Indicator	Categorical Scoring Rubric
Can change the position or direction of the given geometric figure or its subcomponents.	0: Leaves the question unanswered or marks the wrong option. 1: Arrives at the correct answer by making measurements. 2: Arrives at the correct answer by making additions or omissions on the figure.

Categorical Scores	Number of Students	Percentage
0	12	23.53%
1	19	37.25%
2	20	39.22%
	Total=51	Total=100%

As can be seen in the table, 39.22% of the students could use their operative apprehension and compare the given areas. While 37.25% of the remaining students arrived at the correct result by making measurements, 23.53% of them either left the question unanswered or marked the wrong option. This indicates that the majority of the students did not use their operative apprehension.

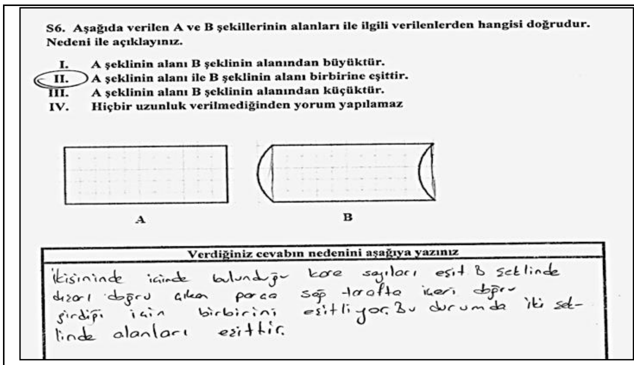
When the written responses of the students (in the 0-point category) who marked the wrong option were examined, it was observed that they arrived at their response by resorting to their perceptual apprehension. One of the written responses given by one of the students who compared the appearance of the figures and based his/her answer on this comparison is as follows:



I think the area of figure A is greater than the area of figure B because they are drawn on the same plane. A takes up more space, while there are empty spaces in B.

Figure 7. The written response of the student who compared the areas based on his/her perceptual apprehension

Stating in his/her written response that there was more space in figure B, when compared to figure A, the student arrived at the conclusion that the area of figure A was greater than that of figure B. This shows that the student's wrong answer can be attributed to his/her being influenced by the visual change deriving from the curved figure. On the other hand, the students in the 1-point category were not influenced by the visual change; they arrived at the result by counting the unit squares. While the unit squares were being counted, even though the unit squares which the sides of the quadrangle divided into pieces were connected, it cannot be affirmed that the process is totally under the control of operative apprehension. Thus, when the written responses of the students in the 2-point category were examined, it was observed that the students focused on the entire figure instead of the unit squares and arrived at a result by decomposing and then recomposing the figure. A written response of a student in the 2-point category and an excerpt from the interview conducted with the same students are as follows:



…As the piece stretching out from figure B moves into the figure from the left, they even out …

Figure 8. The written response of a student in the 2-point category

Researcher: *How did you decide that the areas of figure A and that of figure B in the given question were equal?*

Student: *Here the piece stretching out of figure B fills the empty part and forms figure A. In this way, their areas are equal.*

As can be understood from the given responses, the student arrived at the correct answer by decomposing figure B into its components and changing their positions. In this way, the student overcame the difficulty of counting the unit squares due to the curved parts of side B by resorting to his/her operative apprehension.

Discussion and Conclusion

When the findings which the current study yielded were examined, it was revealed that more than half of the students' perceptual, discursive and operative apprehension were not at the highest level of categorical score. The majority of the students were unsuccessful in recognizing the different geometric shapes within a given geometric figure, in transforming the given verbal information to visual information, producing verbal information based on visual information, drawing conclusions without being influenced by the appearance of the figure, and decomposing and recomposing geometric figures. This indicates that high school students are not at the required level of readiness for their geometry lessons. When students in Turkey begin high school, as of grade 9, they should be able to prove the geometric relationships they had experimentally been affirming in primary and secondary schools (MEB, 2013a, 2018a). As high school geometry classes require students to prove given geometric relationships, students are expected to understand and provide proof in these classes. However, a majority of the students were unsuccessful even in fundamental level conducts as transforming verbal information into visual information. According to Duval (1998), the most important reason underlying this is the importance attached to an increasing level of knowledge in learning environments, while cognitive and apprehension processes are neglected. Thus, secondary school curricula should not only focus on teaching of concepts but should also give place to activities that aim to develop students' figure apprehension processes as mathematics is not a science consisting solely of mathematical outcomes and concepts produced by individuals, but a way of thinking (Cuaco & Goldenberg, 1996). Thus, in numerous studies conducted with high school students, it was revealed that students' proof providing abilities were very weak (Healy & Hoyles, 1998; McCrone & Martin, 2004; Senk, 1985) and this should not be evaluated based solely on students' knowledge of the topic. The current study reveals that it is also important to consider conditions deriving from the structure of students' figure apprehension processes.

One finding that needs to be emphasized in the current study is the fact that a significant percentage of students are still, at high school level, making inferences based on the appearance of figures. This indicates that students' geometric reasoning skills are not at the sufficient level for high school education. While it is regarded acceptable to make decisions based on the appearance of figures at the beginning of secondary school, this approach should be replaced with making decisions based on mathematical principles at the end of secondary school. Thus, according to Fischbein (1993), reasoning based on the appearance of a figure should leave its place to reasoning based on mathematical principles and concepts. Similar findings were reported in other studies as well (Michael, 2013; Ubuz, 1999). To illustrate, Ubuz (1999) reported that students were influenced by the appearance of figures while drawing conclusions as they were not at the required Van Hiele level. Moreover, Michael (2013) accounts for this condition by stating that perceptual apprehension is more dominant in the reasoning process and thus, it affects the other figure apprehension processes. Even though these two explanations are different from each other, in both studies, the problem of making inferences based on the appearance of figures is addressed as a cognitive problem that needs to be overcome. According to Duval (1998), students' tendency to produce results based on perceptual apprehension indicates that they cannot consider the figure mathematically. To consider a figure mathematically, the figure apprehension processes need to be developed independent of each other (Duval, 1998).

According to Duval (1995), the relationship between the structure of the devices used while constructing a geometric figure and the mathematical properties of the figure is an important step in the formation of sequential apprehension. The reason is that while constructing geometric figures, initially the device (isometric paper, compass, ruler, etc.) and subsequently the mathematical properties of the figure are considered, and the figures are constructed based on the relationship built between the two elements. In the present study, students' sequential apprehension was evaluated based on isometric paper. The findings which the study yielded revealed that the sequential apprehension of a significant percentage of the students was within the higher category. However, this should not mean that a similar level of success can be displayed when students utilize different devices (such as a compass and ruler) to construct figures because when the structure of the device used varies, the relationship between the mathematical properties of the geometric figure to be constructed and the structure of the device to be used also varies.

The findings of the present study revealed that more than half of the students' operative apprehension was not at the defined highest level of category. Instead of making inferences by making changes in the original appreance of a given geometric figure, these students made inferences by making measurements or under the influence of the appearance of the figure. This indicates that the students' operative apprehension is not at the expected level. But studies report that changing the original figure and establishing a relationship between geometric figures and mathematical principles are important while solving problem or providing proof (Llinares & Clemente, 2014; Torregrosa & Quesada, 2008). Moreover, in problems where there are similar mathematical properties, but different geometric figures are used, the variations emerging in students' responses where some students immediately recognize the relationship, while others cannot see it are cases that based on the structure of students' operative apprehension (Duval, 1995). On the other hand similar findings also emerged in other studies. Thus, in a study conducted by Michael, Gagathis, Avgerinos & Kuzniak (2011), it was found that students in grades 9 or 10 were least successful in decomposing a figure into its subcomponents and then recomposing them to form different figures, which is an indicator of operative apprehension. In addition, it was observed in the same study that the operative apprehension of 10th grade students was better than that of 9th grade students. This result is consistent with a finding reported by Michael (2013) that figure apprehension processes improve as the grade level increases. However, Michael-Chrysanthou & Gagatis (2013), who examined the operative apprehension of grade 9 and 10 students in their study, reported that there was no significant difference among the student in terms of operative apprehension. Hence, findings from various studies are conflict with each other regarding the finding that students' figure apprehension level increases as their grade level increases. This demonstrates that the development of figure apprehension processes shouldn't be regarded as a natural outcome of learning environments.

As a result, the findings obtained revealed that the 9th grade students' figure apprehension processes were not at a sufficient level when the expected objectives were taken into consideration. This shows that teachers need to focus on not only conceptual knowledge but also the structure of the figure apprehension processes of students prior to geometry classes. Thus, it is believed that the results which the present study yielded are significant as they draw attention to various aspects of geometric thinking. However, to what extent the results of the study can be generalized depends on further studies with different sample groups of students.

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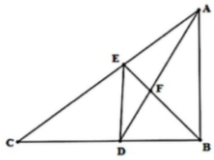
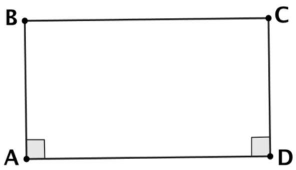
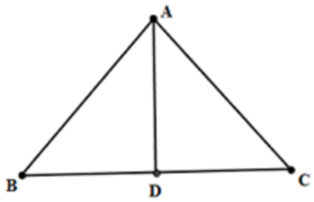
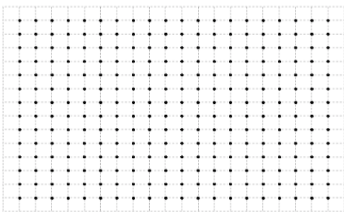
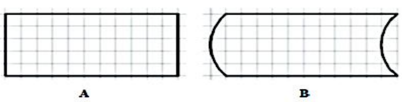
Appendix 1. Figure Apprehension Cognitive Process Test and Categorical Scores

QUESTIONS	CATEGORICAL SCORES
Q1. Which geometrical properties make up the figure given below? Write them down by using the corner points. 	0: Does not mention any figure or leaves the question unanswered. 1: Only writes the triangles or quadrangles. 2: Writes different geometric figures (triangle, quadrangle…).
Q2. By looking at the figure below and the notations on the figure, write the mathematical properties of the figure in the space provided on the right. (You can write more than one feature.) 	0: Other responses: Does not transform the given visual information to verbal information and makes an incorrect inference or leaves the question unanswered. 1: Transforms the given visual information to verbal information accurately but makes an inaccurate inference. 2: Transforms the visual information to verbal information accurately but does not make any inference. 3: Does not transform the given visual information to verbal information but makes an accurate inference. 4: Transforms the given visual information to verbal information accurately and makes correct inferences.
Q3. Use notations to show the given information on the figure below. • $\lvert AB\rvert = \lvert AC\rvert$ • $\lvert AC\rvert = \lvert BC\rvert$ • $[AD] \perp [BC]$ • $m(\hat{C}) = m(\hat{B})$ • $[AD]$, is the angle bisector of angle BAC. • $[AD]$ is a median. 	0: Leaves the question unanswered. 1: Uses correct notations for some verbal information and incorrect ones for others. 2: Uses correct notations but does not show some of the information on the figure. 3: Uses correct notations and shows all the given information on the figure.
Q4. By using the isometric paper given below, draw a parallelogram and a perpendicular line in this parallelogram. 	0: Leaves the question unanswered or cannot accurately draw the parallelogram nor the perpendicular line in the parallelogram. 1: Draws a parallelogram and a perpendicular line in the parallelogram but does not explain how the drawing is done. 2: Draws the parallelogram and the perpendicular line in the parallelogram accurately and makes a correct explanation of how the drawing is done.
Q5. Which of the following statements regarding the areas of figures A and B is correct? Explain why. I. The area of figure A is bigger than that of figure B. II. The area of figure A is equal to that of figure B. III. The area of figure A is smaller than of figure B. IV. No interpretation can be made as no length measurement has been given. 	0: Leaves the question unanswered or chooses the wrong option. 1: Arrives at the right answer by making measurements or chooses the correct option but does not provide an explanation. 2: Arrives at the correct answer by making additions and subtractions.