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AN ALMOST ORTHOSYMMETRIC BILINEAR MAP

RUŞEN YILMAZ

ABSTRACT. In this paper, as a generalization of the concept of pseudo-almost f-algebra, we define a new concept of almost orthosymmetric bilinear map on a vector lattice and prove that the Arens triadjoint of a positive almost orthosymmetric bilinear map is positive almost orthosymmetric. This also extends results on the order bidual of pseudo-almost f-algebras.

1. INTRODUCTION

We studied in [16] a new class of pseudo-almost f-algebra (a lattice ordered algebra A in which $a \wedge b = 0$ in A implies $ab \wedge ba = 0$) and presented its relation with the certain lattice ordered algebras; *f-algebras* [5], *almost f-algebras* [6] and *d-algebras* [12].

In [17], concentrating on the Arens multiplications [2, 3] in the algebraic bidual of pseudo-almost f-algebras (so-called r-algebra in [17]), we prove that the order continuous bidual of an Archimedean pseudo-almost f-algebra is again a Dedekind complete (and hence Archimedean) pseudo-almost f-algebra. This is a generalization of a result of Bernau and Huijsmans in [4] in which they prove that the order continuous bidual of an almost f-algebra (respectively d-algebra) is again an almost f-algebra (respectively d-algebra).

In this paper, as an extension of the notion of pseudo-almost f-algebra, we introduce a new concept of almost orthosymmetric bilinear map and prove that if A, B are vector lattices and $T : A \times A \to B$ is a positive almost orthosymmetric bilinear map, then the triadjoint $T^{***} : (A')'_n \times (A')'_n \to (B')'_n$ is a positive almost orthosymmetric bilinear map. This also generalizes results on the order bidual of pseudo-almost f-algebras in [17].

The Arens multiplication introduced in [3] on the bidual of various lattice ordered algebras has been well documented (see, e.g., [4]). The more general question about Arens triadjoints of bilinear maps on products of vector lattices has recently aroused

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considerable interest (see, e.g., [7]). In this direction, as the extensions of the notions of classes of almost f-algebra, f-algebra, d-algebra and pseudo-almost f-algebra, we have studied the Arens triadjoints of some classes of bilinear maps on vector lattices; mainly, orthosymmetric bilinear maps, bi-orthomorphisms, d-bimorphisms and almost orthomorphism bilinear maps (see [15, 14]):

Definition 1. Let A and B be vector lattices. A bilinear map $T : A \times A \to B$ is said to be
(1) *orthosymmetric* if $x \wedge y = 0$ implies $T(x, y) = 0$ for all $x, y \in A$ (first appeared in a paper by G. Buskes and A. van Rooij in [9] in 2000).
(2) a *bi-orthomorphism* if it is a separately order bounded bilinear map such that $x \wedge y = 0$ in A implies $T(z, x) \wedge y = 0$ for all $z \in A^+$, when $A = B$ (first appears a paper by G. Buskes, R. Page Jr and R. Yilmaz in [10] in 2009).
(3) a *d-bimorphism if* $x \wedge y = 0$ in A implies $T(z, x) \wedge T(z, y) = 0$ for all $z \in A^+$ (first appears in a paper R. Yilmaz in [14] in 2017).
(4) *almost orthosymmetric* if $x \wedge y = 0$ implies $T(x, y) \wedge T(y, x) = 0$ for all $x, y \in A$.

The following theorem is obvious from the above definitions.

Theorem 2. (1) *Every bi-orthomorphism is both orthosymmetric and a d-bimorphism.*
(2) *Every orthosymmetric bilinear map is almost orthosymmetric.*

From here on, let A, B, and C be Archimedean vector lattices and A', B', C' be their respective duals.

A bilinear map $T : A \times B \to C$ can be extended in a natural way to the bilinear map $T^{***} : A'' \times B'' \to C''$ constructed in the following stages:

$$\begin{array}{ll} T^* : C' \times A \to B', & T^*(f, x)(y) = f(T(x, y)) \\ T^{**} : B'' \times C' \to A', & T^{**}(G, f)(x) = G(T^*(f, x)) \\ T^{***} : A'' \times B'' \to C'', & T^{***}(F, G)(f) = F(T^{**}(G, f)) \end{array}$$

for all $x \in A, y \in B, f \in C', F \in A'', G \in B''$ (so-called the *first Arens adjoint* of T).

Another extension of a bilinear map $T : A \times B \to C$ is the map ${}^{***}T : A'' \times B'' \to C''$ constructed in the following stages:

$$\begin{array}{ll} {}^{*}T : B \times C' \to A', & {}^{*}T(y, f)(x) = f(T(x, y)) \\ {}^{**}T : C' \times A'' \to B', & {}^{**}T(f, F)(y) = F({}^{*}T(y, f)) \\ {}^{***}T : A'' \times B'' \to C'', & {}^{***}T(F, G)(f) = G({}^{**}T(f, F)) \end{array}$$

for all $x \in A, y \in B, f \in C', F \in A'', G \in B''$ (so-called the *second Arens adjoint* of T) [3].

In this work we shall concentrate on the first Arens adjoint; that is, we prove that $T^{***} : (A')'_n \times (A')'_n \to (B')'_n$ is positive almost orthosymmetric whenever $T : A \times A \to B$ is so. Similar results hold for the second.

For the elementary theory of vector lattices and terminology not explained here we refer to [1, 13, 18].

2. The triadjoint of an almost orthosymmetric bilinear map

In this section we prove that the extension T^{***} of a positive almost orthosymmetric bilinear map $T : A \times A \to B$ is again positive almost orthosymmetric. We first recall some relevant notions. The *canonical mapping* $a \mapsto \widehat{a}$ of a vector lattice A into its order bidual A'' is defined by $\widehat{a}(f) = f(a)$ for all $f \in A'$. For each $a \in A$, $\widehat{a}$ defines an order continuous algebraic lattice homomorphism on A' and the canonical image $\widehat{A}$ of A is a subalgebra of $(A')'_c$. Moreover the band

$$I_{\widehat{A}} = \{F \in (A')'_c : |F| \leq \widehat{x} \text{ for some } x \in A^+\}$$

generated by $\widehat{A}$ is order dense in $(A')'_c$; i.e., for each $F \in (A')'_c$, there exists an upwards directed net $\{G_\lambda : \lambda \in \Lambda\}$ in $I_{\widehat{A}}$ such that $0 < G_\lambda \uparrow F$.

A bilinear operator $T : A \times B \to C$ is said to be *order bounded* if for all $(x, y) \in A^+ \times B^+$ we have

$$\{T(a, b) : 0 \leq a \leq x, 0 \leq b \leq y\}$$

is order bounded. T is *positive* if for all $x \in A^+$ and $y \in B^+$ we have $T(x, y) \in C^+$. Clearly every positive bilinear map is order bounded. Moreover if T is positive, then so is T^*.

Let $0 \leq f \in B'$ and $x \in A^+$. Then the positive linear functional ${}^*T(x, f)$ in A' defined by, for all $y \in A$,

$${}^*T(x, f)(y) = f(T(y, x))$$

satisfies

$$T^{**}(\widehat{x}, f) =^* T(x, f).$$

Indeed, for all $y \in A$,

$$T^{**}(\widehat{x}, f)(y) = \widehat{x}(T^*(f, y)) = T^*(f, y)(x) = f(T(y, x)) =^* T(x, f)(y).$$

Proposition 3. *Let A, B be vector lattices and $T : A \times A \to B$ be a positive almost orthosymmetric bilinear map. If $x \in A^+$ and $0 \leq G, H \in (A')'_n$ satisfy $G, H \leq \widehat{x}$ and $G \wedge H = 0$, then $T^{***}(G, H) \wedge T^{***}(H, G) = 0$.*

Proof. Let T be positive almost orthosymmetric. Then clearly T^{***} is positive.

Let $0 \leq f \in B'$ and $x \in A^+$. Then $0 \leq {}^*T(x, f) + T^*(f, x) \in A'$, and so, by Corollary 1.2 of [4], there exist $g, h \in A'$ with $g \wedge h = 0$, and $G(g) = 0 = H(h)$ such that

$${}^*T(x, f) + T^*(f, x) = g + h.$$

By the Riesz-Kontorovič Theorem ([1, Theorem 1.13]),

$$\inf\{g(y) + h(z) : x = y + z,\, y, z \in A^+\} = (g \wedge h)(x) = 0,$$

which implies that, for $\epsilon > 0$, there exist $y, z \in A^+$ such that $x = y + z$ and $g(y) < \epsilon$ and $h(z) < \epsilon$.

We now define the linear functionals G_1 and H_1 on A' by

$$G_1 = G \wedge (\widehat{y - y \wedge z}) \qquad \text{and} \qquad H_1 = H \wedge (\widehat{z - y \wedge z}).$$

Clearly, $0 \le G_1, H_1 \in (A')'_c$ and the following inequalities hold.

$$\begin{aligned} 0 \le H - H_1 &= (H - (z - y \wedge z))^+ \le (\widehat{x} - (\widehat{z - y \wedge z}))^+ \\ &= (y + z - \widehat{(z - y \wedge z)})^+ = (\widehat{y + y \wedge z})^+ \le 2\widehat{y}, \qquad (1) \end{aligned}$$

and similarly

$$0 \le G - G_1 \le 2\widehat{z}. \qquad (2)$$

Since T^{***} is positive and $T^{***}(\widehat{a}, \widehat{b}) = \widehat{T(a,b)}$ for all $a, b \in A$, it follows that

$$\begin{aligned} 0 &\le T^{***}(G_1, H_1) \wedge T^{***}(H_1, G_1) \\ &\le T^{***}(\widehat{y - y \wedge z}, \widehat{z - y \wedge z}) \wedge T^{***}(\widehat{z - y \wedge z}, \widehat{y - y \wedge z}) = 0; \end{aligned}$$

$$\text{i.e., } T^{***}(G_1, H_1) \wedge T^{***}(H_1, G_1) = 0. \qquad (3)$$

We next consider the elements

$$0 \le T^{***}(G - G_1, H), T^{***}(G_1, H - H_1), T^{***}(H - H_1, G), T^{***}(H_1, G - G_1)$$

of $(A')'_n$. Then, by the positivity of T^{***} and (1),

$$\begin{aligned} T^{***}(G - G_1, H)(f) &\le T^{***}(G - G_1, \widehat{x})(f) = (G - G_1)(T^{**}(\widehat{x}, f)) \\ &= (G - G_1)(^*T(x, f)) \le (G - G_1)(^*T(x, f) + T^*(f, x)) \\ &= (G - G_1)(g + h) = (G - G_1)(g) + (G - G_1)(h) \\ &\le G(g) + (G - G_1)(h) \le 0 + 2\widehat{z}(h) = 2h(z) \qquad (4) \end{aligned}$$

and, by (2),

$$\begin{aligned} T^{***}(G_1, H - H_1)(f) &\le T^{***}(G, H - H_1)(f) \le T^{***}(\widehat{x}, H - H_1)(f) \\ &= \widehat{x}(T^{**}(H - H_1, f)) = T^{**}(H - H_1, f)(x) \\ &= (H - H_1)(T^*(f, x)) \\ &\le (H - H_1)(T^*(f, x) +^* T(x, f)) = (H - H_1)(g + h) \\ &= (H - H_1)(g) + (H - H_1)(h) \le H(g) + (H - H_1)(h) \\ &\le 0 + 2\widehat{y}(g) = 2g(y). \qquad (5) \end{aligned}$$

It follows by symmetry that

$$T^{***}(H - H_1, G)(f) \le 2g(y) \quad \text{and} \quad T^{***}(H_1, G - G_1)(f) \le 2h(z). \qquad (6)$$

Using the fact that $(a + b) \wedge c \le a \wedge c + b \wedge c \le a + b \wedge c$ in vector lattices and (3), we find

$$\begin{aligned} T^{***}(G, H) \wedge T^{***}(H, G) = (&T^{***}(G - G_1, H) + T^{***}(G_1, H - H_1,) + T^{***}(G_1, H_1)) \\ &\wedge (T^{***}(H - H_1, G) + T^{***}(H_1, G - G_1,) + T^{***}(H_1, G_1)) \\ &\le T^{***}(G - G_1, H) + T^{***}(G_1, H - H_1) \\ &+ T^{***}(G_1, H_1) \wedge (T^{***}(H - H_1, G,) + T^{***}(G_1, G - G_1) \\ &+ T^{***}(H_1, G_1)) \\ &\le T^{***}(G - G_1, H) + T^{***}(G_1, H - H_1) \end{aligned}$$

$$+T^{***}(G_1,H_1)\wedge T^{***}(H-H_1,G)+T^{***}(H_1,G-G_1)$$
$$+T^{***}(G_1,H_1)\wedge T^{***}(H_1,G_1)$$
$$\leq T^{***}(G-G_1,H)+T^{***}(G_1,H-H_1)$$
$$+T^{***}(H-H_1,G)+T^{***}(H_1,G-G_1).$$

Hence, by (4), (5) and (6),

$$0\leq T^{***}(G,H)\wedge T^{***}(H,G)(f)\leq T^{***}(G-G_1,H)(f)+T^{***}(G_1,H-H_1)(f)$$
$$+T^{***}(H-H_1,G)(f)+T^{***}(H_1,G-G_1)(f)$$
$$\leq 2h(z)+2g(y)+2g(y)+2h(z)\leq 8\epsilon.$$

Since this holds for an arbitrary $\epsilon>0$, we have $T^{***}(G,H)\wedge T^{***}(H,G)(f)=0$ for all $0\leq f\in B'$. It now follows that for all $f\in B'$

$$\begin{aligned} T^{***}(G,H)\wedge T^{***}(H,G)(f) &= T^{***}(G,H)\wedge T^{***}(H,G)(f^+)\\ &\quad -T^{***}(G,H)\wedge T^{***}(H,G)(f^-)\\ &= 0, \end{aligned}$$

and so $T^{***}(G,H)\wedge T^{***}(H,G)=0$, as required. □

We are in a position to prove the main result of this paper.

Theorem 4. *Let A,B be vector lattices and $T:A\times A\to B$ be a positive almost orthosymmetric bilinear map. Then the bilinear map $T^{***}:(A')'_n\times(A')'_n\to(B')'_n$ is positive almost orthosymmetric.*

Proof. In the preceding proposition we have proved that the restriction map $T^{***}|_{I_{\widehat{A}}\times I_{\widehat{A}}}$ is positive almost orthosymmetric whenever $T:A\times A\to B$ is so. We now extend the result to the whole $(A')'_n\times A')'_n$. To do this, let $0\leq G,H\in(A')'_n$ such that $G\wedge H=0$. We have to show that $T^{***}(G,H)\wedge T^{***}(H,G)(f)=0$. Since the band $I_{\widehat{A}}$ is order dense in $(A')'_n$, there exist $G_\alpha,H_\beta\in I_{\widehat{A}}$ such that $0\leq G_\alpha\uparrow G$ and $0\leq H_\beta\uparrow H$ with $0\leq G_\alpha\leq\widehat{x}_\alpha$ and $0\leq H_\beta\leq\widehat{y}_\beta$ for some $x_\alpha,y_\beta\in A^+$. It follows from $G\wedge H=0$ that $G_\alpha\wedge H_\beta=0$ for all α,β. Furthermore, $0\leq G_\alpha,H_\beta\leq\widehat{x_\alpha+y_\beta}$. Hence, by above, we see that

$$T^{***}(G_\alpha,H_\beta)\wedge T^{***}(H_\beta,G_\alpha)=0 \tag{7}$$

for all α and β. Now let $0\leq f\in B'$. It follows from $0\leq H_\beta\uparrow H$ that $0\leq H_\beta(T^{**}(f,x))\uparrow H(T^{**}(f,x))$;

$$\text{i.e., } 0\leq T^{**}(H_\beta,f)(x)\uparrow T^{**}(H,f)(x)$$

for all $0\leq x\in A$. This shows that $0\leq T^{**}(H_\beta,f)\uparrow T^{**}(H,f)$. Hence, by the order continuity of G_α for each α, $0\leq G_\alpha(T^{**}(H_\beta,f))\uparrow G_\alpha(T^{**}(H,f))$;

$$\text{i.e., } 0\leq T^{***}(G_\alpha,H_\beta)(f)\uparrow T^{***}(G_\alpha,H)(f)$$

which implies that, for each α,

$$0\leq T^{***}(G_\alpha,H_\beta)\uparrow T^{***}(G_\alpha,H). \tag{8}$$

Similarly, since $0 \leq G_\alpha \uparrow G$, we have $0 \leq G_\alpha(T^{**}(H,f)) \uparrow G(T^{**}(H,f))$;

$$\text{i.e., } 0 \leq T^{***}(G_\alpha, H)(f) \uparrow T^{***}(G,H)(f)$$

for all $0 \leq f \in B'$, and so

$$0 \leq T^{***}(G_\alpha, H) \uparrow T^{***}(G,H) \tag{9}$$

In the same way, by the order continuity of H_β for each β, we obtain

$$0 \leq T^{***}(H_\beta, G_\alpha) \uparrow T^{***}(H_\beta, G) \tag{10}$$

leading to

$$0 \leq T^{***}(H_\beta, G_\alpha) \uparrow T^{***}(H,G). \tag{11}$$

Now it follows from (8) and (10) that

$$0 \leq T^{***}(G_\alpha, H_\beta) \wedge T^{***}(H_\beta, G_\alpha) \uparrow T^{***}(G_\alpha, H) \wedge T^{***}(H_\beta, G),$$

and so, by (7),

$$T^{***}(G_\alpha, H) \wedge T^{***}(H_\beta, G) = 0 \tag{12}$$

for all α, β. On the other hand, from (9) and (11) we have

$$0 \leq T^{***}(G_\alpha, H) \wedge T^{***}(H_\beta, G) \uparrow T^{***}(G,H) \wedge T^{***}(H,G).$$

It follows from (12) that

$$T^{***}(G,H) \wedge T^{***}(H,G) = 0,$$

as required. □

As the Arens multiplications are separately order continuous and in a commutative algebra a pseudo-almost f-algebra and almost f-algebra coincide, we immediately obtain the following corollary.

Corollary 5. (1) *The order continuous bidual of a pseudo-almost f-algebra is a Dedekind complete (and hence Archimedean) pseudo-almost f-algebra.*
(2) *The order bidual of a commutative pseudo-almost f-algebra is a Dedekind complete pseudo-almost f-algebra.*

Another way of obtaining the result of Proposition 3 is by means of the approximation by components ([11]). First we observe some notations: Let A be a vector lattice and let a be a fixed element of A. If $E := \{F \in (A')'_n : \exists \lambda > 0, |F| \leq \lambda \widehat{a}\}$-the ideal generated in $(A')'_n$ by $\widehat{a}$. Consider the Boolean algebra $\mathcal{R}$ generated by the set of all band projections of E onto principal bands generated by positive elements of $\widehat{A}$ in E. If we denote the band projection onto the band generated in E by the element $F \in E$ by P_F, then $\mathcal{R}$ is generated by the set $\mathcal{G} := \{P_{\widehat{x}} : x \in A^+\}$-the set of all band projections onto the principal ideals generated by elements $\widehat{x}$ with $x \in A^+$. Also, $\mathcal{G}\widehat{a} := \{P_{\widehat{x}}\widehat{a} : x \in A^+\}$.

Proposition 6. *Let A, B be vector lattices and $T : A \times A \to B$ be a positive almost orthosymmetric bilinear map. If $x \in A^+$ and $0 \leq G, H \in (A')'_n$ satisfy $G, H \leq \widehat{x}$ and $G \wedge H = 0$ (that is, G and H are two disjoint elements of the band $I_{\widehat{A}} = \{F \in (A')'_n : |F| \leq \widehat{x}$ for some $x \in A^+\}$ generated by $\widehat{A}$, which is order dense in $(A')'_n$), then $T^{***}(G, H) \wedge T^{***}(H, G) = 0$.*

Proof. It is sufficient to proof that $T^{***}(P_G\widehat{x}, P_H\widehat{x}) \wedge T^{***}(P_H\widehat{x}, P_G\widehat{x}) = 0$ since $0 \leq G \leq P_G\widehat{x}$ and $0 \leq H \leq P_H\widehat{x}$. (Note that, as band projections are positive, $0 \leq G\wedge H = P_G G \wedge P_H H \leq P_G\widehat{x} \wedge P_H\widehat{x}$, and so $P_G\widehat{x} \wedge P_H\widehat{x} = 0$ implies $G \wedge H = 0$. Hence $T^{***}(G, H) \wedge T^{***}(H, G) \leq T^{***}(P_G\widehat{x}, P_H\widehat{x}) \wedge T^{***}(P_H\widehat{x}, P_G\widehat{x})$ by the positivity of T^{***}.) But, to do this, it is sufficient to proof that $T^{***}(\widehat{x}-F, F) \wedge T^{***}(F, \widehat{x}-F) = 0$ for any component F of $\widehat{x}$; that is, $\widehat{x} - F \wedge F = 0$.

The proof of this is in four steps, as follows.

Step 1. Let $F \in \mathcal{G}\widehat{a}$, say $F = P_{\widehat{a}}\widehat{x} = \sup_n(n\widehat{a} \wedge \widehat{x})$. Then it follows from

$$\widehat{x} - F = \widehat{x} - \sup_n(n\widehat{a} \wedge \widehat{x}) = \inf_n(\widehat{x} - n\widehat{a} \wedge \widehat{x}) = \inf_n(\widehat{x} - n\widehat{a})^+$$

and that for each fixed n

$$\begin{aligned} 0 &\leq T^{***}(\widehat{x} - F, (n\widehat{a} - \widehat{x})^+) \wedge T^{***}((n\widehat{a} - \widehat{x})^+, \widehat{x} - F) \\ &\leq T^{***}((\widehat{x} - n\widehat{a})^+, (n\widehat{a} - \widehat{x})^+) \wedge T^{***}((\widehat{x} - n\widehat{a})^+, (n\widehat{a} - \widehat{x})^+) \\ &= T((x - n\widehat{a)^+, (n}a - x)^+) \wedge T((x - n\widehat{a)^+, (n}a - x)^+) \\ &= T((x - na)^+, (na - x)\widehat{^+) \wedge T}((x - na)^+, (na - x)^+) \\ &= 0, \end{aligned}$$

as $(x - na)^+ \wedge (na - x)^+ = 0$ and T is almost orthosymmetric (where we use the fact that $T^{***}(\widehat{a}, \widehat{b}) = \widehat{T(a, b)}$ for all $a, b \in A$). Hence

$$T^{***}(\widehat{x} - F, (n\widehat{a} - \widehat{x})^+) \wedge T^{***}((n\widehat{a} - \widehat{x})^+, \widehat{x} - F) = 0,$$

and so

$$n(T^{***}(\widehat{x} - F, (n\widehat{a} - \widehat{x})^+ \wedge T^{***}((n\widehat{a} - \widehat{x})^+, \widehat{x} - F))) = 0.$$

This implies that for each n

$$T^{***}(\widehat{x} - F, (\widehat{a} - \frac{1}{n}\widehat{x}))^+ \wedge T^{***}((\widehat{a} - \frac{1}{n}\widehat{x})^+, \widehat{x} - F) = 0.$$

Therefore

$$T^{***}(\widehat{x} - F, \widehat{a}) \wedge T^{***}(\widehat{a}, \widehat{x} - F) = 0, \quad \text{as} \quad n \to \infty.$$

It follows that for each n

$$n(T^{***}(\widehat{x}-F, \widehat{a}) \wedge T^{***}(\widehat{a}, \widehat{x}-F)) = 0; \quad \text{i.e., } T^{***}(\widehat{x}-F, n\widehat{a}) \wedge T^{***}(n\widehat{a}, \widehat{x}-F) = 0.$$

Hence,

$$0 \leq T^{***}(\widehat{x}-F, n\widehat{a}\wedge\widehat{x}) \wedge T^{***}(n\widehat{a}\wedge\widehat{x}, \widehat{x}-F) \leq T^{***}(\widehat{x}-F, n\widehat{a}) \wedge T^{***}(n\widehat{a}, \widehat{x}-F) = 0;$$

$$\text{i.e., } T^{***}(\widehat{x} - F, n\widehat{a} \wedge \widehat{x}) \wedge T^{***}(n\widehat{a} \wedge \widehat{x}, \widehat{x} - F) = 0.$$

Since this holds for each n, we get

$$\sup_n (T^{***}(\widehat{x}-F, n\widehat{a}\wedge\widehat{x}) \wedge T^{***}(n\widehat{a}\wedge\widehat{x}, \widehat{x}-F)) = 0,$$

which leads that, by the separately order continuity of T^{***} (since T is positive, T is of order bounded variation, and so T^{***} is separately order continuous (see e.g. Theorem 2.1 in [7])),

$$\begin{aligned} 0 &\leq T^{***}(\widehat{x}-F, F) \wedge T^{***}(F, \widehat{x}-F) \\ &= T^{***}(\widehat{x}-F, \sup_n (n\widehat{a}\wedge\widehat{x})) \wedge T^{***}(\sup_n (n\widehat{a}\wedge\widehat{x}), \widehat{x}-F) \\ &= \sup_n (T^{***}(\widehat{x}-F, n\widehat{a}\wedge\widehat{x})) \wedge \sup_n (T^{***}((n\widehat{a}\wedge\widehat{x}), \widehat{x}-F)) \\ &= \sup_n (T^{***}(\widehat{x}-F, n\widehat{a}\wedge\widehat{x}) \wedge T^{***}((n\widehat{a}\wedge\widehat{x}), \widehat{x}-F)) \\ &= 0; \end{aligned}$$

$$\text{i.e., } T^{***}(\widehat{x}-F, F) \wedge T^{***}(F, \widehat{x}-F) = 0.$$

Step 2. Let $F = \bigwedge_{i=1}^m F_i$ where either $F_i \in \mathcal{G}\widehat{a}$ or $\widehat{x} - F_i \in \mathcal{G}\widehat{a}$. Then

$$\widehat{x} - F = \bigvee_{i=1}^m (\widehat{x} - F_i),$$

and so

$$\begin{aligned} 0 &\leq T^{***}(\widehat{x}-F, F) \wedge T^{***}(F, \widehat{x}-F) \\ &= T^{***}(\bigvee_{i=1}^m (\widehat{x}-F_i), \bigwedge_{i=1}^m F_i) \wedge T^{***}(\bigwedge_{i=1}^m F_i, \bigvee_{i=1}^m (\widehat{x}-F_i)) \\ &\leq T^{***}(\sum_{i=1}^m (\widehat{x}-F_i), F_i) \wedge T^{***}(F_i, \sum_{i=1}^m (\widehat{x}-F_i)) \\ &= \sum_{i=1}^m T^{***}((\widehat{x}-F_i), F_i) \wedge \sum_{i=1}^m T^{***}(F_i, \widehat{x}-F_i) \\ &\leq \sum_{i=1}^m (T^{***}((\widehat{x}-F_i), F_i) \wedge T^{***}(F_i, \widehat{x}-F_i)) \\ &= 0 \qquad \text{(by Step 1)}; \end{aligned}$$

$$\text{i.e., } T^{***}(\widehat{x}-F, F) \wedge T^{***}(F, \widehat{x}-F) = 0.$$

Step 3. Let $F = \bigvee_{i=1}^n F_i$ where each F_i is of the form F had in Step 1 (that is, $F_i = \bigwedge_{j=1}^m F_{ij}, \forall i = 1, 2, \cdots, n$, and so $F = \bigvee_{i=1}^n \bigwedge_{j=1}^m F_{ij}$). Then, in the same way as Step 2,

$$\widehat{x} - F = \bigwedge_{i=1}^m (\widehat{x} - F_i),$$

and so

$$\begin{aligned}
0 &\le T^{***}(\widehat{x}-F,F)\wedge T^{***}(F,\widehat{x}-F)\\
&= T^{***}(\bigwedge_{i=1}^{m}(\widehat{x}-F_i),\bigvee_{i=1}^{m}F_i)\wedge T^{***}(\widehat{x}-F_i,\bigvee_{i=1}^{m}F_i)\\
&\le T^{***}(\widehat{x}-F_i,\bigvee_{i=1}^{m}F_i)\wedge T^{***}(\widehat{x}-F_i,\bigvee_{i=1}^{m}F_i)\\
&\le T^{***}(\widehat{x}-F_i,\sum_{i=1}^{m}F_i)\wedge T^{***}(\sum_{i=1}^{m}F_i,\widehat{x}-F_i)\\
&= \sum_{i=1}^{m}(T^{***}(\widehat{x}-F_i,F_i))\wedge \sum_{i=1}^{m}(T^{***}(F_i,\widehat{x}-F_i))\\
&\le \sum_{i=1}^{m}(T^{***}(\widehat{x}-F_i,F_i)\wedge T^{***}(F_i,\widehat{x}-F_i))\\
&= 0 \qquad \text{(by Step 2)};
\end{aligned}$$

$$\text{i.e., } T^{***}(\widehat{x}-F,F)\wedge T^{***}(F,\widehat{x}-F)=0.$$

Step 4. Let $F\in\mathcal{R}\widehat{x}$. If $F=\sup_\alpha F_\alpha$ or $F=\inf_\alpha F_\alpha$ with each F_α is a component of $\widehat{x}$ (that is, $(\widehat{x}-F_\alpha)\wedge F_\alpha=0$ for each α) having the property that

$$T^{***}(\widehat{x}-F_\alpha,F_\alpha)\wedge T^{***}(F_\alpha,\widehat{x}-F_\alpha)=0,$$

then using the separately order continuity of T^{***} we show that F has the same property;

$$\text{i.e., } T^{***}(\widehat{x}-F,F)\wedge T^{***}(F,\widehat{x}-F)=0.$$

Indeed, suppose that $F=\sup_\alpha F_\alpha$. For each fixed α and for all $\beta\ge\alpha$ we have $F_\beta\ge F_\alpha$, and so $\widehat{x}-F_\beta\le\widehat{x}-F_\alpha$. Hence, by the positivity of T^{***} and the hypothesis,

$$0\le T^{***}(\widehat{x}-F_\beta,F_\alpha)\wedge T^{***}(F_\alpha,\widehat{x}-F_\beta)\le T^{***}(\widehat{x}-F_\alpha,F_\alpha)\wedge T^{***}(F_\alpha,\widehat{x}-F_\alpha)=0;$$

$$\text{i.e., } T^{***}(\widehat{x}-F_\beta,F_\alpha)\wedge T^{***}(F_\alpha,\widehat{x}-F_\beta)=0 \quad \forall\beta\ge\alpha.$$

Therefore

$$\inf_{\beta\ge\alpha}(T^{***}(\widehat{x}-F_\beta,F_\alpha)\wedge T^{***}(F_\alpha,\widehat{x}-F_\beta))=0,$$

and so, by the order continuity of lattice operations ($x_\tau\downarrow x$ and $y_\tau\downarrow y$ implies $x_\tau\wedge y_\tau\downarrow x\wedge y$),

$$\inf_{\beta\ge\alpha}T^{***}(\widehat{x}-F_\beta,F_\alpha)\wedge\inf_{\beta\ge\alpha}T^{***}(F_\alpha,\widehat{x}-F_\beta)=0.$$

Since T^{***} is a separately order continuous,

$$T^{***}(\inf_{\beta\ge\alpha}(\widehat{x}-F_\beta),F_\alpha)\wedge T^{***}(F_\alpha,\inf_{\beta\ge\alpha}(\widehat{x}-F_\beta))=0;$$

$$\text{i.e., } T^{***}(\widehat{x}-F,F_\alpha)\wedge T^{***}(F_\alpha,\widehat{x}-F)=0.$$

Since this holds for all α,

$$\sup_{\alpha}(T^{***}(\widehat{x}-F,F_\alpha)\wedge T^{***}(F_\alpha,\widehat{x}-F))=0,$$

from which it follows that

$$T^{***}(\widehat{x}-F,F)\wedge T^{***}(F,\widehat{x}-F)=0,$$

by the order continuity of lattice operations (if $x_\tau \uparrow x$ and $y_\tau \uparrow y$, then $x_\tau \wedge y_\tau \uparrow x\wedge y$), as above.

In exactly the same way above we now show that if $F=\inf_\alpha F_\alpha$ such that $(\widehat{x}-F_\alpha)\wedge F_\alpha=0$ and $T^{***}(\widehat{x}-F_\alpha,F_\alpha)\wedge T^{***}(F_\alpha,\widehat{x}-F_\alpha)=0$ for each α, then

$$T^{***}(\widehat{x}-F,F)\wedge T^{***}(F,\widehat{x}-F)=0.$$

Let α be fixed. Then we have $F_\beta\le F_\alpha$ for all $\beta\ge\alpha$. Hence, by the positivity of T^{***} and the hypothesis,

$$0\le T^{***}(\widehat{x}-F_\alpha,F_\beta)\wedge T^{***}(F_\beta,\widehat{x}-F_\alpha)\le T^{***}(\widehat{x}-F_\alpha,F_\alpha)\wedge T^{***}(F_\alpha,\widehat{x}-F_\alpha)=0;$$

$$\text{i.e., } T^{***}(\widehat{x}-F_\alpha,F_\beta)\wedge T^{***}(F_\beta,\widehat{x}-F_\alpha)=0,\qquad \forall\beta\ge\alpha.$$

Therefore

$$\inf_{\beta\ge\alpha}(T^{***}(\widehat{x}-F_\alpha,F_\beta)\wedge T^{***}(F_\beta,\widehat{x}-F_\alpha))=0,$$

and so,

$$\inf_{\beta\ge\alpha}T^{***}(\widehat{x}-F_\alpha,F_\beta)\wedge\inf_{\beta\ge\alpha}T^{***}(F_\beta,\widehat{x}-F_\alpha)=0.$$

Since T^{***} is a separately order continuous,

$$T^{***}(\widehat{x}-F_\alpha,\inf_{\beta\ge\alpha}F_\beta)\wedge T^{***}(\inf_{\beta\ge\alpha}F_\beta,\widehat{x}-F_\alpha)=0.$$

$$\text{i.e., } T^{***}(\widehat{x}-F_\alpha,F)\wedge T^{***}(F,\widehat{x}-F_\alpha)=0.$$

Since this holds for all α, we get

$$\sup_{\alpha}(T^{***}(\widehat{x}-F_\alpha,F)\wedge T^{***}(F,\widehat{x}-F_\alpha))=0.$$

Therefore

$$T^{***}(\widehat{x}-F,F)\wedge T^{***}(F,\widehat{x}-F)=0,$$

from which the result follows. □

We conclude our work with the following important remark for further research.

Remark. The triadjoints on the whole order biduals is still an open problem. One has to obtain a way to handle the singular parts of order biduals, as the cases of orthosymmetric bilinear maps and bi-orthomorphisms [15], in order to prove that the triadjoint $T^{***}:A''\times A''\to B''$ of an almost orthosymmetric bilinear map $T:A\times A\to B$ is an almost orthosymmetric bilinear map.

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References

[1] Aliprantis, C. D. and Burkinshaw, O., Positive Operators, Academic Press, 1985.
[2] Arens, R., Operations induced in function classes, *Monatsh. Math.,* 55, (1951), 1-19.
[3] Arens, R., The adjoint of a bilinear operation, *Proc. Amer. Math. Soc.,* 2, (1951), 839-848.
[4] Bernau, S. J. and Huijsmans, C. B., The order bidual of almost f-algebras and d-algebras, *Trans. Amer. Math. Soc.,* 347, (1995), 4259-4275.
[5] Birkhoff, G. and Pierce, R. S., Lattice-ordered rings, *An. Acad. Brasil. Ciénc.,* 28, (1956), 41-49.
[6] Birkhoff, G., Lattice Theory, Amer. Math. Soc. Colloq. Publ. 25, 1967.
[7] Boulabiar, K., Buskes G. and Pace, R., Some properties of bilinear maps of order bounded variation, *Positivity,* 9, (2005), 401-414.
[8] Kusraev, A. G., Representation and extension of orthoregular bilinear operators, *Vladikavkaz Mat. Zh.,* 9, (2007), 16-29.
[9] Buskes, G. and van Rooij, A., Almost f-algebras: commutativity and Cauchy-Schwarz inequality, *Positivity*, 4, (2000), 227-231.
[10] Buskes, G., Page, R. Jr. and Yilmaz, R., A note on bi-orthomorphisms, *Operator Theory: Advances and Applications,* 201, (2009), 99-107.
[11] Grobler, J. J., Commutativity of Arens product in lattice ordered algebras, *Positivity,* 3, (1999), 357-364.
[12] Kudláček, V., On some types of ℓ-rings, *Sborni Vysokého Učeni Techn v Brně,* 1-2, (1962), 179-181.
[13] Luxemburg, W. A. J. and Zaanen, A. C., Riesz Spaces I, North-Holland, 1971.
[14] Yilmaz, R., A Note on bilinear maps on vector lattices, *New Trends in Mathematical Sciences,* 5 (3), (2017), 168-174.
[15] Yılmaz, R.,The Arens triadjoints of some bilinear maps, *Filomat*, 28 (5), (2014), 963-979.
[16] Yilmaz, R., Notes on lattice ordered algebras, *Serdica Math. J.,* 40, (2014), 319-328.
[17] Yilmaz, R., The bidual of r-algebras, *Ukrainian Mathematical Journal,* 63 (5), (2011), 833-837.
[18] Zaanen, A. C., Introduction to Operator Theory in Riesz Spaces, Springer, 1997.

Current address: Ruşen Yılmaz: Department of Mathematics, Faculty of Arts and Science, Recep Tayyip Erdogan University, 53100 Rize, Turkey.

E-mail address: `rusen.yilmaz@erdogan.edu.tr`

ORCID Address: `https://orcid.org/0000-0003-1579-2234`