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THE BINOMIAL ALMOST CONVERGENT AND NULL SEQUENCE SPACES

MUSTAFA CEMİL BİŞGİN

Abstract. In this paper, we introduce the sequence spaces $f(B^{r,s})$, $f_0(B^{r,s})$ and $fs(B^{r,s})$ which generalize the Kirişçi's work [16]. Moreover, we show that these spaces are BK-spaces and are linearly isomorphic to the sequence spaces f, f_0 and fs, respectively. Furthermore, we mention the Schauder basis and give β, γ-duals of these spaces. Finally, we determine some matrix classes related to these spaces.

1. Introduction

The family of all real(or complex) valued sequences is a vector space under usual coordinate-wise addition and scalar multiplication and is denoted by w. Every vector subspace of w is called a sequence space. The notations of ℓ_∞, c_0, c and ℓ_p are used for the spaces of all bounded, null, convergent and absolutely p-summable sequences, respectively, where $1 \leq p < \infty$.

A BK-space is a Banach sequence space provided each of the maps $p_i : X \longrightarrow \mathbb{C}$, $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$, where X is a sequence space. According to this definition, the sequence spaces ℓ_∞, c_0 and c are BK-spaces with their sup-norm defined by $\|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n|$ and ℓ_p is a BK-space with its ℓ_p-norm defined by

$$\|x\|_{\ell_p} = \left(\sum_{k=0}^{\infty} |x_k|^p \right)^{\frac{1}{p}}$$

where $1 \leq p < \infty$ [2].

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Let $A = (a_{nk})$ be an infinite matrix of complex numbers. For any $x = (x_k) \in w$, the A-transform of x is written by $y = Ax$ and is defined by

$$y_n = (Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k \tag{1.1}$$

for all $n \in \mathbb{N}$ and each of these series being assumed convergent [3]. For simplicity in notation, we henceforth prefer that the summation without limits runs from 0 to ∞.

Given two arbitrary sequence spaces X and Y, the class of all matrices $A = (a_{nk})$ such that $Ax \in Y$ for all $x \in X$ is denoted by $(X : Y)$.

The domain of an infinite matrix $A = (a_{nk})$ in a sequence space X is denoted by X_A defined by

$$X_A = \{x = (x_k) : Ax \in X\} \tag{1.2}$$

which is also a sequence space. The domain of summation matrix $S = (s_{nk})$ in sequence spaces c and ℓ_∞ are called the spaces of all convergent and bounded series and are denoted by cs and bs, respectively, where $S = (s_{nk})$ is defined by

$$s_{nk} = \begin{cases} 1 & , \quad 0 \le k \le n \\ 0 & , \quad k > n \end{cases}$$

for all $n, k \in \mathbb{N}$.

A matrix is called a triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n, k \in \mathbb{N}$. Also, a triangle matrix $A = (a_{nk})$ uniquely has an inverse A^{-1} such that A^{-1} is a triangle matrix.

As an application of the Hahn-Banach theorem to the sequence space ℓ_∞, the notion of Banach Limits was first introduced by the Stefan Banach. Banach first recognized certain non-negative linear functionals on ℓ_∞ which remain invariant under shift operators and which are extension of l, where $l : c \longrightarrow \mathbb{R}$, $l(x) = \lim_{n\to\infty} x_n$ is defined and l is linear functional on c. Such functionals were later termed "Banach Limits" [1].

A functional $L : \ell_\infty \longrightarrow \mathbb{R}$ is called a Banach Limit if the following conditions hold

(i) $L(ax_n + by_n) = aL(x_n) + bL(y_n) \quad a, b \in \mathbb{R}$

(ii) $L(x_n) \geq 0$ if $x_n \geq 0, \quad n = 0, 1, 2, ...$

(iii) $L(P_j(x_n)) = L(x_n), \quad P_j(x_n) = x_{n+j}, \quad j = 1, 2, 3, ...$

(iv) $L(e) = 1$ where $e = (1, 1, ...)$

Lorentz continued the study of Banach Limits and brought out a new concept called Almost Convergence. The bounded sequence $x = (x_n)$ is called almost convergent and the number $Lim x_n = \lambda$ is called its F-limit if $L(x_n) = \lambda$ holds for every limit L [4].

The theory of matrix transformation has a great importance in the theory of summability which was obtained by Cesàro, Norlund, Borel, Riesz... . Therefore, many authors have constructed new sequence spaces by using matrix domain of infinite matrices. For instance, $(\ell_\infty)_{N_q}$ and c_{N_q} in [5], X_p and X_∞ in [6], e_0^r and e_c^r in [7], e_p^r and e_∞^r in [8] and [9], $e_0^r(\Delta), e_c^r(\Delta)$ and $e_\infty^r(\Delta)$ in [10], $e_0^r(\Delta^m), e_c^r(\Delta^m)$ and $e_\infty^r(\Delta^m)$ in [11], $e_0^r(B^{(m)}), e_c^r(B^{(m)})$ and $e_\infty^r(B^{(m)})$ in [12], $e_0^r(\Delta,p), e_c^r(\Delta,p)$ and $e_\infty^r(\Delta,p)$ in [13], $\hat{f}_0$ and $\hat{f}$ in [14], $f_0(B)$ and $f(B)$ in [15], $f_0(E)$ and $f(E)$ in [16].

In this paper, we introduce the sequence spaces $f(B^{r,s})$, $f_0(B^{r,s})$ and $fs(B^{r,s})$ which generalize the Kirişçi's work [16]. Moreover, we show that these spaces are BK-spaces and are linearly isomorphic to the sequence spaces f, f_0 and fs, respectively. Furthermore, we mention the Schauder basis and give β, γ-duals of these spaces. Finally, we determine some matrix classes related to these spaces.

2. The Binomial Almost Convergent And Null Sequence Spaces

In this part, we give some historical informations and define the sequence spaces $f_0(B^{r,s})$, $f(B^{r,s})$ and $fs(B^{r,s})$. Furthermore, we show that these spaces are BK-spaces and are linearly isomorphic to the sequence spaces f_0, f and fs, respectively.

Lorentz obtained the following characterization for almost convergent sequences.

Theorem 1 (see [4])**.** *In order that F-limit, $Lim x_n = \lambda$ exists for the sequence $x = (x_n)$, it is necessary and sufficient that*

$$\lim_{k\to\infty} \frac{x_n + x_{n+1} + ... + x_{n+k}}{k+1} = \lambda$$

holds uniformly in n.

By taking into account the notion of almost convergence and Theorem 1, the space of all almost convergent sequences, almost null sequences and almost convergent series are defined by

$$f = \left\{ x = (x_k) \in w : \exists \lambda \in \mathbb{C} \ni \lim_{i\to\infty} \sum_{k=0}^{i} \frac{x_{n+k}}{i+1} = \lambda \text{ uniformly in } n \right\},$$

$$f_0 = \left\{ x = (x_k) \in w : \lim_{i\to\infty} \sum_{k=0}^{i} \frac{x_{n+k}}{i+1} = 0 \text{ uniformly in } n \right\},$$

and

$$fs = \left\{ x = (x_k) \in w : \exists \lambda \in \mathbb{C} \ni \lim_{i\to\infty} \sum_{k=0}^{i} \sum_{j=0}^{n+k} \frac{x_j}{i+1} = \lambda \text{ uniformly in } n \right\},$$

respectively.

By considering the notion of (1.2), the sequence space fs can be rearranged by means of the summation matrix $S = (s_{nk})$ as follows:

$$fs = f_S \tag{2.1}$$

Theorem 2 (see [17]). *The inclusions $c \subset f \subset \ell_\infty$ strictly hold.*

Theorem 3 (see [17]). *The sequence spaces f and f_0 are BK-spaces with the norm $\|x\|_f = \sup_{i,n\in\mathbb{N}} \left|\sum_{k=0}^{i} \frac{x_{n+k}}{i+1}\right|$ and fs is a BK-space with the norm $\|x\|_{fs} = \|Sx\|_f$.*

In order to define sequence spaces, the Euler matrix was first considered by Altay, Başar and Mursaleen in [7], [8] and [9]. They constructed the Euler sequence spaces e_0^r, e_c^r, e_∞^r and e_p^r as follows:

$$e_0^r = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k = 0 \right\},$$

$$e_c^r = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k \ \text{ exists} \right\},$$

$$e_\infty^r = \left\{ x = (x_k) \in w : \sup_{n\in\mathbb{N}} \left| \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k \right| < \infty \right\}$$

and

$$e_p^r = \left\{ x = (x_k) \in w : \sum_{n=0}^{\infty} \left| \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k \right|^p < \infty \right\}.$$

where $1 \le p < \infty$, and the Euler matrix $E^r = (e_{nk}^r)$ is defined by

$$e_{nk}^r = \begin{cases} \binom{n}{k}(1-r)^{n-k} r^k & , \quad 0 \le k \le n \\ 0 & , \quad k > n \end{cases}$$

for all $n, k \in \mathbb{N}$, where $0 < r < 1$.

Afterward, Kirişçi used the Euler matrix in [16] for defining Euler almost null and Euler almost convergent sequence spaces. These spaces are defined by

$$f_0(E) = \left\{ x = (x_k) \in w : \lim_{m\to\infty} \sum_{j=0}^{m} \sum_{k=0}^{n+j} \frac{\binom{n+j}{k}(1-r)^{n+j-k} r^k x_k}{m+1} = 0 \text{ uniformly in } n \right\}$$

and

$$f(E) = \left\{ x = (x_k) \in w : \exists \lambda \in \mathbb{C} \ni \lim_{m\to\infty} \sum_{j=0}^{m} \sum_{k=0}^{n+j} \frac{\binom{n+j}{k}(1-r)^{n+j-k} r^k x_k}{m+1} = \lambda \text{ uniformly in } n \right\}.$$

Recently, Bişgin has defined the Binomial sequence spaces $b_0^{r,s}$, $b_c^{r,s}$, $b_\infty^{r,s}$ and $b_p^{r,s}$ in [18] and [19] as follows:

$$b_0^{r,s} = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k = 0 \right\},$$

$$b_c^{r,s} = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k \text{ exists} \right\},$$

$$b_\infty^{r,s} = \left\{ x = (x_k) \in w : \sup_{n\in\mathbb{N}} \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k \right| < \infty \right\}$$

and

$$b_p^{r,s} = \left\{ x = (x_k) \in w : \sum_n \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k \right|^p < \infty \right\}$$

where $1 \leq p < \infty$ and the Binomial matrix $B^{r,s} = (b_{nk}^{r,s})$ is defined by

$$b_{nk}^{r,s} = \begin{cases} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k & , \quad 0 \leq k \leq n \\ 0 & , \quad k > n \end{cases}$$

for all $k, n \in \mathbb{N}$, $r, s \in \mathbb{R}$ and $rs > 0$. Here, we would like to touch on a point, if we take $r + s = 1$, we obtain the Euler sequence spaces e_0^r, e_c^r, e_∞^r and e_p^r. Therefore Bişgin has generalized the Altay, Başar and Mursaleen's works.

Now, we define the sequence spaces $f_0(B^{r,s})$, $f(B^{r,s})$ and $fs(B^{r,s})$ by

$$f_0(B^{r,s}) = \left\{ x = (x_k) \in w : \lim_{i\to\infty} \sum_{j=0}^{i} \sum_{k=0}^{n+j} \frac{\binom{n+j}{k} s^{n+j-k} r^k x_k}{(i+1)(r+s)^{n+j}} = 0 \text{ uniformly in } n \right\},$$

$$f(B^{r,s}) =$$

$$\left\{ x = (x_k) \in w : \exists \lambda \in \mathbb{C} \ni \lim_{i\to\infty} \sum_{j=0}^{i} \sum_{k=0}^{n+j} \frac{\binom{n+j}{k} s^{n+j-k} r^k x_k}{(i+1)(r+s)^{n+j}} = \lambda \text{ uniformly in } n \right\}$$

and

$$fs(B^{r,s}) =$$

$$\left\{ x = (x_k) \in w : \exists \lambda \in \mathbb{C} \ni \lim_{i\to\infty} \sum_{j=0}^{i} \sum_{\nu=0}^{n+j} \sum_{k=0}^{\nu} \frac{\binom{\nu}{k} s^{\nu-k} r^k x_k}{(i+1)(r+s)^{\nu}} = \lambda \text{ uniformly in } n \right\},$$

respectively. By taking into account the notation (1.2), the sequence spaces $f_0(B^{r,s})$, $f(B^{r,s})$ and $fs(B^{r,s})$ can be redefined by means of the domain of the Binomial matrix $B^{r,s} = (b_{nk}^{r,s})$ as follows:

$$f_0(B^{r,s}) = (f_0)_{B^{r,s}}, \; f(B^{r,s}) = f_{B^{r,s}} \text{ and } fs(B^{r,s}) = fs_{B^{r,s}} \tag{2.2}$$

In addition, given an arbitrary sequence $x = (x_k) \in w$, the $B^{r,s}$-transform of $x = (x_k)$ is defined by

$$y_k = (B^{r,s}x)_k = \frac{1}{(s+r)^k} \sum_{j=0}^{k} \binom{k}{j} s^{k-j} r^j x_j \tag{2.3}$$

for all $k \in \mathbb{N}$.

Theorem 4. *The sequence spaces $f_0(B^{r,s})$, $f(B^{r,s})$ and $fs(B^{r,s})$ endowed with the norms*

$$\|x\|_{f(B^{r,s})} = \|x\|_{f_0(B^{r,s})} = \|B^{r,s}x\|_f \text{ and } \|x\|_{fs(B^{r,s})} = \|B^{r,s}x\|_{fs}$$

are BK-spaces, respectively.

Proof. We know that f, f_0 and fs are BK-spaces. Also, $B^{r,s} = (b^{r,s}_{nk})$ is a triangle matrix and the condition (2.2) holds. By combining these three facts and Theorem 4.3.12 of Wilansky[3], we deduce that $f(B^{r,s})$, $f_0(B^{r,s})$ and $fs(B^{r,s})$ are BK-spaces. This completes the proof. □

Theorem 5. *The sequence spaces $f_0(B^{r,s})$, $f(B^{r,s})$ and $fs(B^{r,s})$ are linearly isomorphic to the sequence spaces f_0, f and fs, respectively.*

Proof. Since the relations $f_0(B^{r,s}) \cong f_0$ and $fs(B^{r,s}) \cong fs$ can be shown by using a similar way, we give the proof of theorem for only the sequence space $f(B^{r,s})$. For this, we should show the existence of a linear bijection between the sequence spaces $f(B^{r,s})$ and f.

Let us consider the transformation $L : f(B^{r,s}) \longrightarrow f$ such that $L(x) = B^{r,s}x$. Then it is obvious that for every $x = (x_k) \in f(B^{r,s})$, $L(x) = B^{r,s}x \in f$. Moreover, it is clear that L is a linear transformation and $x = 0$ whenever $L(x) = 0$. Because of this, L is injective.

Now, we define a sequence $x = (x_k)$ by means of the sequence $y = (y_k) \in f$ by

$$x_k = \frac{1}{r^k}\sum_{j=0}^{k}\binom{k}{j}(-s)^{k-j}(s+r)^j y_j$$

for all $k \in \mathbb{N}$. Then, we have

$$\begin{aligned}(B^{r,s}x)_k &= \frac{1}{(s+r)^k}\sum_{j=0}^{k}\binom{k}{j}s^{k-j}r^j x_j \\ &= \frac{1}{(s+r)^k}\sum_{j=0}^{k}\binom{k}{j}s^{k-j}\sum_{i=0}^{j}\binom{j}{i}(-s)^{j-i}(s+r)^i y_i \\ &= y_k\end{aligned}$$

for all $k \in \mathbb{N}$. This shows us that

$$\lim_{i\to\infty}\sum_{j=0}^{i}\sum_{k=0}^{n+j}\frac{\binom{n+j}{k}s^{n+j-k}r^k x_k}{(i+1)(r+s)^{n+j}} = \lim_{i\to\infty}\sum_{j=0}^{i}\frac{y_{n+j}}{i+1} = F - lim y_n$$

namely, $x = (x_k) \in f(B^{r,s})$ and $L(x) = y$. Therefore L is surjective. Moreover, for all $x = (x_k) \in f(B^{r,s})$, we know that

$$\|L(x)\|_f = \|B^{r,s}x\|_f = \|x\|_{f(B^{r,s})}$$

So, L is norm preserving. As a results of these, L is a linear bijection which says us that the sequence space $f(B^{r,s})$ is linearly isomorphic to the sequence space f, that is $f(B^{r,s}) \cong f$. This completes the proof. $\square$

Theorem 6. *The inclusion $c \subset f(B^{r,s})$ is strict.*

Proof. It is obvious that the inclusion $c \subset f(B^{r,s})$ holds. Now, we consider the sequence $x = (x_k)$ defined by $x_k = (-1)^k$ for all $k \in \mathbb{N}$. Then, $x = (x_k) \notin c$ but $B^{r,s}x = \left(\left(\frac{s-r}{s+r}\right)^k\right) \in f$, namely $x \in f(B^{r,s})$. So, the inclusion $c \subset f(B^{r,s})$ strictly holds. This completes the proof. $\square$

3. The Schauder Basis And β, γ-Duals

In this part, we speak of the Schauder basis and give β, γ-duals of the spaces $f(B^{r,s})$ and $fs(B^{r,s})$.

Let us start with the definition of the Schauder basis. For a given normed space $(X, \|.\|_X)$, a sequence $b = (b_k)$ of elements of X is called a Schauder basis for X, if and only if, for all $x \in X$, there exists a unique sequence $\mu = (\mu_k)$ of scalar such that $x = \sum_k \mu_k b_k$; i.e. such that

$$\left\| x - \sum_{k=0}^{n} \mu_k b_k \right\|_X \longrightarrow 0$$

as $n \to \infty$.

Corollary 1 (see [14])**.** *Almost convergent sequence space f has no Schauder basis.*

Remark 1. *For an arbitrary sequence space X and a triangle matrix $A = (a_{nk})$, it is known that X_A has a basis if and only if X has a basis* [20]*.*

By combining this fact and Corollary 1, we can give the next result.

Corollary 2. *The sequence spaces $f(B^{r,s})$ and $fs(B^{r,s})$ have no Schauder basis.*

The multiplier space of two arbitrary sequence spaces X and Y is defined by

$$M(X,Y) = \Big\{ a = (a_k) \in w : xa = (x_k a_k) \in Y \text{ for all } \ x = (x_k) \in X \Big\}$$

By using this definition and sequence spaces cs and bs, the β- and γ-duals of a sequence space X are defined by

$$X^\beta = M(X, cs) \text{ and } X^\gamma = M(X, bs)$$

respectively.

Now, we give some statements which are used in the next lemma. Let $A = (a_{nk})$ be an infinite matrix of complex numbers.

$$\sup_{n \in \mathbb{N}} \sum_k |a_{nk}| < \infty \tag{3.1}$$

$$\lim_{n\to\infty} a_{nk} = \alpha_k \text{ for each fixed } \; k \in \mathbb{N} \tag{3.2}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = \alpha \tag{3.3}$$

$$\lim_{n\to\infty} \sum_k |\Delta(a_{nk} - \alpha_k)| = 0 \tag{3.4}$$

$$\sup_{n\in\mathbb{N}} \sum_k |\Delta a_{nk}| < \infty \tag{3.5}$$

$$\lim_{k\to\infty} a_{nk} = 0 \text{ for each fixed } \; n \in \mathbb{N} \tag{3.6}$$

$$\lim_{n\to\infty} \sum_k |\Delta^2 a_{nk}| = \alpha \tag{3.7}$$

where $\Delta a_{nk} = a_{nk} - a_{n,k+1}$ and $\Delta^2 a_{nk} = \Delta(\Delta a_{nk})$.

Lemma 1. *For an infinite matrix $A = (a_{nk})$, the following statements hold:*

(i) $A = (a_{nk}) \in (f : \ell_\infty) \Leftrightarrow$ (3.1) *holds (see* [21]*)*
(ii) $A = (a_{nk}) \in (f : c) \Leftrightarrow$ (3.1), (3.2), (3.3) *and* (3.4) *hold (see* [21]*)*
(iii) $A = (a_{nk}) \in (fs : \ell_\infty) \Leftrightarrow$ (3.5) *and* (3.6) *hold (see* [14]*)*
(iv) $A = (a_{nk}) \in (fs : c) \Leftrightarrow$ (3.2), (3.5), (3.6) *and* (3.7) *hold (see* [22]*)*

Theorem 7. *Given the sets $t_1^{r,s}$, $t_2^{r,s}$, $t_3^{r,s}$, $t_4^{r,s}$, $t_5^{r,s}$, $t_6^{r,s}$ and $t_7^{r,s}$ as follows:*

$$t_1^{r,s} = \left\{ a = (a_k) \in w : \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \binom{j}{k} (-s)^{j-k} (r+s)^k r^{-j} a_j \right| < \infty \right\}$$

$$t_2^{r,s} = \left\{ a = (a_k) \in w : \lim_{n\to\infty} \sum_{j=k}^{n} \binom{j}{k} (-s)^{j-k} (r+s)^k r^{-j} a_j \text{ exists for each } \; k \in \mathbb{N} \right\}$$

$$t_3^{r,s} = \left\{ a = (a_k) \in w : \lim_{n\to\infty} \sum_{k=0}^{n} \left[\sum_{j=0}^{k} \binom{k}{j} (-s)^{k-j} (r+s)^j r^{-k} \right] a_k \text{ exists} \right\}$$

$$t_4^{r,s} = \left\{ a = (a_k) \in w : \lim_{n\to\infty} \sum_k \left| \Delta \left[\sum_{j=k}^{n} \binom{j}{k} (-s)^{j-k} (r+s)^k r^{-j} a_j - \alpha_k \right] \right| = 0 \right\}$$

$$t_5^{r,s} = \left\{ a = (a_k) \in w : \sup_{n\in\mathbb{N}} \sum_k \left| \Delta \left[\sum_{j=k}^{n} \binom{j}{k} (-s)^{j-k} (r+s)^k r^{-j} a_j \right] \right| < \infty \right\}$$

$$t_6^{r,s} = \left\{ a = (a_k) \in w : \lim_{k\to\infty} \sum_{j=k}^{n} \binom{j}{k} (-s)^{j-k} (r+s)^k r^{-j} a_j = 0 \;\; \text{for each } \; n \in \mathbb{N} \right\}$$

and

$$t_7^{r,s} = \left\{ a = (a_k) \in w : \lim_{n\to\infty} \sum_k \left| \Delta^2 \left[\sum_{j=k}^{n} \binom{j}{k} (-s)^{j-k} (r+s)^k r^{-j} a_j \right] \right| \text{ exists} \right\}$$

where $\lim_{n\to\infty} \sum_{j=k}^{n} \binom{j}{k} (-s)^{j-k} (r+s)^k r^{-j} a_j = \alpha_k$ for all $k \in \mathbb{N}$.

Then, the following statements hold.

(i) $\{f(B^{r,s})\}^\beta = t_1^{r,s} \cap t_2^{r,s} \cap t_3^{r,s} \cap t_4^{r,s}$
(ii) $\{f(B^{r,s})\}^\gamma = t_1^{r,s}$
(iii) $\{fs(B^{r,s})\}^\beta = t_2^{r,s} \cap t_5^{r,s} \cap t_6^{r,s} \cap t_7^{r,s}$
(iv) $\{fs(B^{r,s})\}^\gamma = t_5^{r,s} \cap t_6^{r,s}$

Proof. To avoid the repetition of similar statements, the proof of theorem is given for only part (i). For any $a = (a_k) \in w$, we consider the sequence $x = (x_k)$ defined by

$$x_k = \frac{1}{r^k} \sum_{j=0}^{k} \binom{k}{j} (-s)^{k-j} (r+s)^j y_j$$

for all $k \in \mathbb{N}$. Then, we get

$$\begin{aligned} \sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} \left[\frac{1}{r^k} \sum_{j=0}^{k} \binom{k}{j} (-s)^{k-j} (r+s)^j y_j \right] a_k \\ &= \sum_{k=0}^{n} \left[\sum_{j=k}^{n} \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_j \right] y_k \\ &= \left(D^{r,s} y \right)_n \end{aligned}$$

for all $n \in \mathbb{N}$, where the matrix $D^{r,s} = (d_{nk}^{r,s})$ is defined by

$$d_{nk}^{r,s} = \begin{cases} \sum_{j=k}^{n} \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_j & , \quad 0 \le k \le n \\ 0 & , \quad k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. So, $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in f(B^{r,s})$ if and only if $D^{r,s} y \in c$ whenever $y = (y_k) \in f$. This gives us that $a = (a_k) \in \{f(B^{r,s})\}^\beta$ if and only if $D^{r,s} \in (f : c)$. By combining this and Lemma 1 (ii), we obtain that $a = (a_k) \in \{f(B^{r,s})\}^\beta$ if and only if

$$\sup_{n\in\mathbb{N}} \sum_k |d_{nk}^{r,s}| < \infty,$$

$$\lim_{n\to\infty} d_{nk}^{r,s} = \alpha_k \text{ for each fixed } k \in \mathbb{N},$$

$$\lim_{n\to\infty}\sum_k d_{nk}^{r,s}=\alpha$$

and

$$\lim_{n\to\infty}\sum_k |\Delta(d_{nk}^{r,s}-\alpha_k)|=0.$$

As a consequence $\{f(B^{r,s})\}^\beta = t_1^{r,s}\cap t_2^{r,s}\cap t_3^{r,s}\cap t_4^{r,s}$. This completes the proof. □

4. Matrix Classes

In this part, we determine some matrix classes related to the sequence spaces $f(B^{r,s})$ and $fs(B^{r,s})$.

For simplicity of notation, from now on, we use the following connections.

$$g_{nk}^{r,s}=\sum_{j=k}^{\infty}\binom{j}{k}(-s)^{j-k}r^{-j}(r+s)^k a_{nj} \tag{4.1}$$

$$h_{nk}^{r,s}=\frac{1}{(s+r)^n}\sum_{j=0}^{n}\binom{n}{j}s^{n-j}r^j a_{jk} \tag{4.2}$$

for all $n,k\in\mathbb{N}$, respectively.

Theorem 8. *For a given sequence space X, assume that the infinite matrices $A=(a_{nk})$, $G^{r,s}=(g_{nk}^{r,s})$ and $H^{r,s}=(h_{nk}^{r,s})$ are connected with the relations* (4.1) *and* (4.2)*. Then, the following statements hold.*

(i) *$A\in(f(B^{r,s}):X)\Leftrightarrow G^{r,s}\in(f:X)$ and $\{a_{nk}\}_{k\in\mathbb{N}}\in\{f(B^{r,s})\}^\beta$ for all $n\in\mathbb{N}$,*
(ii) *$A\in(X:f(B^{r,s}))\Leftrightarrow H^{r,s}\in(X:f)$.*

Proof. (i) We suppose that $A\in(f(B^{r,s}):X)$. By considering the fact that $f(B^{r,s})$ and f are linearly isomorphic, we take an arbitrary sequence $y=(y_k)\in f$, where $y=B^{r,s}x$. Then, $G^{r,s}B^{r,s}$ exists and $\{a_{nk}\}_{k\in\mathbb{N}}\in\{f(B^{r,s})\}^\beta$ for all $n\in\mathbb{N}$. This gives us that $\{g_{nk}^{r,s}\}_{k\in\mathbb{N}}\in\ell_1$ for each $n\in\mathbb{N}$. Thus, $G^{r,s}y$ exists and

$$\sum_k g_{nk}^{r,s}y_k=\sum_k a_{nk}x_k$$

for all $n\in\mathbb{N}$, namely $G^{r,s}y=Ax$. So, $G^{r,s}\in(f:X)$.

Conversely, we suppose that $G^{r,s}\in(f:X)$ and $\{a_{nk}\}_{k\in\mathbb{N}}\in\{f(B^{r,s})\}^\beta$ for all $n\in\mathbb{N}$. Let us take an arbitrary sequence $x=(x_k)\in f(B^{r,s})$. Then, it is clear that

Ax exists. Also, we have

$$\begin{aligned}\sum_{k=0}^{\sigma} a_{nk}x_k &= \sum_{k=0}^{\sigma}\left[\frac{1}{r^k}\sum_{j=0}^{k}\binom{k}{j}(-s)^{k-j}(r+s)^j y_j\right]a_{nk}\\ &= \sum_{k=0}^{\sigma}\left[\sum_{j=k}^{\sigma}\binom{j}{k}(-s)^{j-k}r^{-j}(r+s)^k a_{nj}\right]y_k\end{aligned}$$

for all $n \in \mathbb{N}$. By passing to limit as $\sigma \to \infty$, we deduce that $Ax = G^{r,s}y$. This leads us $A \in (f(B^{r,s}) : X)$.

(ii) For any $x = (x_k) \in X$, we consider the following equality:

$$\begin{aligned}\{B^{r,s}(Ax)\}_n &= \frac{1}{(r+s)^n}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}r^k(Ax)_k\\ &= \sum_k \frac{1}{(r+s)^n}\sum_{j=0}^{n}\binom{n}{j}s^{n-j}r^j a_{jk}x_k\\ &= \{H^{r,s}x\}_n\end{aligned}$$

for all $n \in \mathbb{N}$. By going to the generalized limit, we obtain that $Ax \in f(B^{r,s})$ if and only if $H^{r,s}x \in f$. This completes the proof. □

Now, we list some properties in order to give next lemma. Let $A = (a_{nk})$ be an infinite matrix of complex numbers.

$$F - \lim_{n\to\infty} a_{nk} = \alpha_k \text{ for all fixed } k \in \mathbb{N} \tag{4.3}$$

$$F - \lim_{n\to\infty}\sum_k a_{nk} = \alpha \tag{4.4}$$

$$F - \lim_{n\to\infty}\sum_{j=0}^{n} a_{jk} = \alpha_k \text{ for all fixed } k \in \mathbb{N} \tag{4.5}$$

$$\sup_{n\in\mathbb{N}}\sum_k\left|\Delta\Big(\sum_{j=0}^{n} a_{jk}\Big)\right| < \infty \tag{4.6}$$

$$\sup_{n\in\mathbb{N}}\sum_k\left|\sum_{j=0}^{n} a_{jk}\right| < \infty \tag{4.7}$$

$$\sum_n a_{nk} = \alpha_k \textit{ for all fixed } k \in \mathbb{N} \tag{4.8}$$

$$\sum_n\sum_k a_{nk} = \alpha \tag{4.9}$$

$$\lim_{n\to\infty}\sum_k\left|\Delta\Big[\sum_{j=0}^{n} a_{jk} - \alpha_k\Big]\right| = 0 \tag{4.10}$$

$$\lim_{\vartheta\to\infty}\sum_k\Big|\frac{1}{\vartheta+1}\sum_{j=0}^{\vartheta}a_{n+j,k}-\alpha_k\Big|=0 \text{ uniformly in } n \tag{4.11}$$

$$\lim_{\vartheta\to\infty}\sum_k\Big|\Delta\Big[\frac{1}{\vartheta+1}\sum_{j=0}^{\vartheta}a_{n+j,k}-\alpha_k\Big]\Big|=0 \text{ uniformly in } n \tag{4.12}$$

$$\lim_{\vartheta\to\infty}\sum_k\frac{1}{\vartheta+1}\Big|\sum_{i=0}^{\vartheta}\Delta\Big[\sum_{j=0}^{n+i}a_{jk}-\alpha_k\Big]\Big|=0 \text{ uniformly in } n \tag{4.13}$$

$$\lim_{\vartheta\to\infty}\sum_k\frac{1}{\vartheta+1}\Big|\sum_{i=0}^{\vartheta}\Delta^2\Big[\sum_{j=0}^{n+i}a_{jk}-\alpha_k\Big]\Big|=0 \text{ uniformly in } n \tag{4.14}$$

Lemma 2. *Let $A=(a_{nk})$ be an infinite matrix of complex numbers. Then, the followings hold:*

(i) $A=(a_{nk})\in(c:f)\Leftrightarrow$ (3.1), (4.3) *and* (4.4) *hold (see* [23]*)*
(ii) $A=(a_{nk})\in(\ell_\infty:f)\Leftrightarrow$ (3.1), (4.3) *and* (4.11) *hold (see* [24]*)*
(iii) $A=(a_{nk})\in(f:f)\Leftrightarrow$ (3.1), (4.3), (4.4) *and* (4.12) *hold (see* [24]*)*
(iv) $A=(a_{nk})\in(f:cs)\Leftrightarrow$ (4.7), (4.8), (4.9) *and* (4.10) *hold (see* [26]*)*
(v) $A=(a_{nk})\in(cs:f)\Leftrightarrow$ (3.5) *and* (4.3) *hold (see* [25]*)*
(vi) $A=(a_{nk})\in(cs:fs)\Leftrightarrow$ (4.5) *and* (4.6) *hold (see* [25]*)*
(vii) $A=(a_{nk})\in(bs:f)\Leftrightarrow$ (3.5), (3.6), (4.3) *and* (4.13) *hold (see* [27]*)*
(viii) $A=(a_{nk})\in(bs:fs)\Leftrightarrow$ (3.6), (4.5), (4.6) *and* (4.13) *hold (see* [27]*)*
(ix) $A=(a_{nk})\in(fs:f)\Leftrightarrow$ (3.6), (4.3), (4.12) *and* (4.13) *hold (see* [28]*)*
(x) $A=(a_{nk})\in(fs:fs)\Leftrightarrow$ (4.5), (4.6), (4.13) *and* (4.14) *hold (see* [28]*)*

By combining Lemma 1, relations (4.1), (4.2), Theorem 8 and Lemma 2, the following results can be given.

Corollary 3. *Let us replace the entries of the matrix $A=(a_{nk})$ by those of the matrix $G^{r,s}=(g_{nk}^{r,s})$ in* (3.1)-(3.7) *and* (4.3)-(4.14)*, then the followings hold:*

(i) $A=(a_{nk})\in(f(B^{r,s}):c)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}}\in\{f(B^{r,s})\}^\beta$ *for all* $n\in\mathbb{N}$ *and* (3.1), (3.2), (3.3) *and* (3.7) *hold.*
(ii) $A=(a_{nk})\in(f(B^{r,s}):\ell_\infty)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}}\in\{f(B^{r,s})\}^\beta$ *for all* $n\in\mathbb{N}$ *and* (3.1) *holds.*
(iii) $A=(a_{nk})\in(f(B^{r,s}):cs)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}}\in\{f(B^{r,s})\}^\beta$ *for all* $n\in\mathbb{N}$ *and* (4.7), (4.8), (4.9) *and* (4.10) *hold.*
(iv) $A=(a_{nk})\in(f(B^{r,s}):bs)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}}\in\{f(B^{r,s})\}^\beta$ *for all* $n\in\mathbb{N}$ *and* (4.8) *holds.*

Corollary 4. *Let us replace the entries of the matrix $A=(a_{nk})$ by those of the matrix $H^{r,s}=(h_{nk}^{r,s})$ in* (3.1)-(3.7) *and* (4.3)-(4.14)*, then the followings hold:*

(i) $A=(a_{nk})\in(c:f(B^{r,s}))\Leftrightarrow$ (3.1), (4.3) *and* (4.4) *hold,*
(ii) $A=(a_{nk})\in(\ell_\infty:f(B^{r,s}))\Leftrightarrow$ (3.1), (4.3) *and* (4.11) *hold,*

(iii) $A = (a_{nk}) \in (f : f(B^{r,s})) \Leftrightarrow$ (3.1), (4.3), (4.4) *and* (4.12) *hold,*
(iv) $A = (a_{nk}) \in (cs : f(B^{r,s})) \Leftrightarrow$ (3.5) *and* (4.3) *hold,*
(v) $A = (a_{nk}) \in (bs : f(B^{r,s})) \Leftrightarrow$ (3.5), (3.6), (4.3) *and* (4.13) *hold,*
(vi) $A = (a_{nk}) \in (fs : f(B^{r,s})) \Leftrightarrow$ (3.6), (4.3), (4.12) *and* (4.13) *hold,*
(vii) $A = (a_{nk}) \in (cs : fs(B^{r,s})) \Leftrightarrow$ (4.5) *and* (4.6) *hold,*
(viii) $A = (a_{nk}) \in (bs : fs(B^{r,s})) \Leftrightarrow$ (3.6), (4.5), (4.6) *and* (4.13) *hold,*
(ix) $A = (a_{nk}) \in (fs : fs(B^{r,s})) \Leftrightarrow$ (4.5), (4.6), (4.13) *and* (4.14) *hold.*

5. Conclusion

By taking into account the definition of the Binomial matrix $B^{r,s} = (b^{r,s}_{nk})$, we deduce that $B^{r,s} = (b^{r,s}_{nk})$ reduces in the case $r + s = 1$ to the $E^r = (e^r_{nk})$ which is called the method of Euler means of order r. So, our results obtained from the matrix domain of the Binomial matrix $B^{r,s} = (b^{r,s}_{nk})$ are more general and more extensive than the results on the matrix domain of the Euler means of order r. Moreover, the Binomial matrix $B^{r,s} = (b^{r,s}_{nk})$ is not a special case of the weighed mean matrices. So, the paper fills up a gap in the existent literature.

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Current address: Mustafa CEMİL BİŞGİN: Recep Tayyip Erdoğan University, Faculty Of Arts And Sciences, Department of Mathematics, Zihni Derin Campus, 53100 RİZE/TURKEY.

E-mail address: mustafa.bisgin@erdogan.edu.tr