A semi-Markovian renewal reward process with $\Gamma(g)$ distributed demand

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Abstract: We consider a classical semi-Markovian stochastic model of type (s, S) with Logistic distributed demand random variables. Logistic distribution is a member of special distribution class known as $\Gamma(g)$ that encounters in many real-life applications involving extreme value theory. The objective of this study is to observe some major characteristics of a stochastic process $X(t)$ which represents semi-Markovian renewal reward process of type (s, S). We used new approximation results for renewal function that allow us to obtain three-term asymptotic expansion for ergodic distribution function and for n^{th} order moments of ergodic distribution of the process $X(t)$.

Key words: Logistic distribution, $\Gamma(g)$ class of distributions, heavy tailed distributions, semi-Markovian inventory model, ergodic distribution

1. Introduction
The current study is motivated by an interest in observation of some characteristics of a semi-Markovian stochastic model of type (s, S) when demand random variables are in a special class known as the class of $\Gamma(g)$. The class of $\Gamma(g)$ include examples such as Weibull, logistic, and extreme value distributions that are frequently encountered in extreme value theory. A source of inspiration for this work is the study by Mitov and Omey [18], where they examined lots of specific features of the class of $\Gamma(g)$. Moreover, they provide an approximation for renewal function for this class and showed their approximation cover many of the known results in the literature.

Investigation of semi-Markovian systems are one of the important problems related to stock control. In semi-Markovian stochastic models, interrenewal periods are usually expressed using random variables. Hence, examination of characteristics of semi-Markovian stochastic models are closely related to investigation of renewal function. Let us introduce some notations in order to further clarify the relationship between renewal function with considered system and to formulate the problem precisely.

Suppose a company plans to create the optimal inventory policy and the process $X(t)$ represents the stock level at this company's depot at random time t. Assume that S represents the maximum capacity of this depot and initial stock level at time $t = 0$,; hence,

$$X(0) = X_0 = S.$$

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Suppose that this system must meet incoming demands $\{\eta_n\}_{n\geq 1}$ at random times $T_1, T_2, ..., T_n$. Here $\{\eta_n\}_{n\geq 1}$ random variables are independent and identically distributed (i.i.d.). Moreover, let $\{\xi_n\}_{n\geq 1}$ are i.i.d random variables of inter arrival times between two successive demands. The sequences defined below are known as renewal sequences:

$$T_n = \sum_{i=1}^{n} \xi_i, \ S_n = \sum_{i=1}^{n} \eta_i. \tag{1.1}$$

The stock level $X(t)$ decreases by $\eta_1, \eta_2, ..., \eta_n$ until it decreases below a predetermined inventory control level s at the random time τ_1. Here τ_1 is the first time that the stock level falls below the control level. In this case the change in stock would be as follows:

$$X(T_1) \equiv X_1 = S - \eta_1, \ X(T_2) \equiv X_2 = S - (\eta_1 + \eta_2), ..., X(T_n) \equiv X_n = S - \sum_{i=1}^{n} \eta_i$$

When the stock falls below a certain level s, it is filled up to the level S by external intervention and the first period is completed. The second period starts with a new initial level S and continues in a similar manner to the first period. This process is considered a classical semi-Markovian inventory model of type (s, S) and different modifications of this system is also available. There is an extensive body of literature on this system, of which we will now present a survey.

Khaniyev and Atalay [15] considered a semi-Markovian inventory model of type (s, S) with triangular distributed interference of chance and proved weak convergence theorem for ergodic distribution of the process $X(t)$. Khaniyev and Aksop [14] used generalized Beta distribution for the discrete interference of chance and obtained asymptotic expansion for ergodic distribution function. Aliyev et al. [4] deal with this system with triangular distributed interference of chance and provided asymptotic expansions for the moments of ergodic distribution function. In the study by Aliyev et al. [2], inventory model of type (s,S) has been considered with general interference of chance and asymptotic expansion for n^{th} order moments are obtained. Khaniyev and Mammadova [16] considered the extended model of type (s, S) with Gaussian distribution of summands and investigated some stationary characteristics of this system. Moreover, Hanalioglu et al. [12] studied renewal-reward process with a normal distributed interference of chance and they obtained three-term asymptotic expansion for n^{th} order moments of considered system. In the most classical studies on semi-Markovian inventory models mentioned above it has been assumed that the demand random variables come from probability distribution that has moments of any order which is typically violated in practice. In fact, some studies regarding inventory control models have provided real-life examples on existence of heavy tailed demand distributions. Thus, in recent years, investigating the stock control models with heavy tailed demand quantities theoretically and analyzing their effects have become an important field of study. It is well known that heavy tailed distributions have different kind of tail behavior. Thus, when investigating the systems expressed through renewal function, asymptotic expansion of renewal function should be considered by acknowledging the different tail behavior of each subclasses. The first studies on investigation of semi-Markovian inventory models with heavy tailed distribution were performed by Aliyev [1, 3], Bektaş et al. [6, 7], and Kesemen et al. [17]. In Kesemen et al. [17] and Aliyev et al. [3] a semi-Markovian inventory model of type (s, S) with discrete interference of chance and a classical model have been considered with subexponential demands, respectively. Furthermore, asymptotic expansion for ergodic distribution function of the process $X(t)$ and asymptotic expansion for the moments of ergodic distribution were obtained. Bektaş et al. [8] took $L \cap D$

subclass of heavy tailed distributions for demand random variables and provided asymptotic expansion for ergodic distribution function of the process. Besides, in publications by Bektaş et al. [6, 7] semi-Markovian inventory model of type (s, S) is considered with general class of regularly varying distributions with infinite variance, i.e. when $L(x)$ slowly varies they consider demand random variables with following tail distribution:

$$\bar{F}(x) = \mathbb{P}(\eta_1 \geq x) = x^{-\alpha} L(x), \ 1 < \alpha < 2.$$

They obtained asymptotic results for ergodic distribution function of the process $X(t)$ and moments for ergodic distribution using a special asymptotic expansion for renewal function provided by Geluk [11] respectively.

There is a rich literature on approximation of renewal function with heavy tailed components. The above mentioned studies have been made possible with the help of the articles regarding asymptotic expansions for renewal function generated by different subclass of heavy tailed distributions. Geluk and Frenk [9] obtained two-term asymptotic expansion when demand random variables are subexponential. Anderson and Athreya [5] considered renewal function of regularly varying random variables with infinite variance and provided some asymptotic results. Geluk [11] took the results of Anderson and Athreya [5] one step further and obtained asymptotic expansion for renewal function generated by regularly varying random variables with infinite variance. For more recent work we refer the interested reader to [9–11, 18] and the references therein. The special cases discussed so far in the literature (subexponential and regularly varying case) do not include examples such as extreme value distributions, gamma-type, and logistic distributions which are the most important distributions of the extreme value theory. These distributions are members of a special class of distributions which is called the class of $\Gamma(g)$. In the study by Geluk and de Haan [10] extended class of $\Gamma(g)$ is defined in the following way:

Definition 1.1 *A positive and measurable function* h *belongs to the class of* $\Gamma(g)$ *with auxiliary function* g *if and only if every fixed* $y \in \mathbb{R}$

$$lim_{x \to \infty} h(x + yg(x))/h(x) = e^{-y}. \tag{1.2}$$

Here the auxiliary function g *in (1.2) should satisfy*

$$g(x) = o(x), \ x \to \infty.$$

Mitov and Omey [18] investigated some asymptotic properties of the class of $\Gamma(g)$. When tail distribution belongs to the extended class of $\Gamma(g)$, i.e. $\bar{F}(x) = \mathbb{P}\{\eta_1 > x\} \in \Gamma(g)$ they also provided intuitive approximation for the renewal function generated by the random variables $\{\eta_n\}_{n \geq 1}$ as follows:

$$U(x) = \frac{x}{\mu_1} + \frac{\mu_e}{\mu_1} - \frac{1}{\mu_1^2} g^2(x) \bar{F}(x). \tag{1.3}$$

Here $\mu_i = E(\eta^i)$ for $i = 1, 2$. Moreover, when $\mu_1 < \infty$, $\mu_e = \int_0^\infty (1 - F_e(x)) dx$ where F_e denotes the equilibrium distribution defined as

$$F_e(x) = \frac{1}{\mu_1} \int_0^x \bar{F}(y) dy.$$

It is clear that when $\mu_2 < \infty$, then $\mu_e < \infty$; hence, the following statement is the same as statement (1.3):

$$U(x) = \frac{x}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - \frac{1}{\mu_1^2} g^2(x) \bar{F}(x). \tag{1.4}$$

In this study, our aim is to investigate classical semi-Markovian inventory model of type (s, S) when demand random variables are in the extended subclass of $\Gamma(g)$. Specifically, we use logistic distributed random variables which represents the demand quantities. The logistic distribution has been used for growth models and in a certain type of regression known as the logistic regression. This distribution has also wide application in population modeling, survival analysis, geological sciences, inventory systems and so on. The logistic distribution is mainly used in different areas from different disciplines because the curve has a simple cumulative distribution formula and approximates the normal distribution extremely well. The bell shape of the logistic distribution and the normal distribution are very similar but logistic distribution tends to have heavier tail than normal distribution. It is well known that finding cumulative probabilities for the normal distribution usually involves looking up values in the z-table. Exact values are usually found with statistical software, because the cumulative distribution function is so difficult to work with, involving complicated integration. Hence, some other functions should be used to approximate normal distribution in practice. While there are many functions that can approximate the normal distribution, they also tend to have very complicated mathematical formulas. The logistic distribution, in comparison, has a much simpler cumulative distribution formula i.e.:

$$\bar{F}(x) = \frac{C}{1 + e^x}$$

One of the important properties of this study is that we consider semi-Markovian inventory model of type (s, S) for the first time with a $\Gamma(g)$ distributed component which is an important subclass of heavy tailed distributions. Specifically, we used logistic distribution that approximates normal distribution very well. Moreover, in the literature two-term asymptotic expansions provided for ergodic distribution function and moments for ergodic distribution so far. Assuming that demand random variables have logistic distribution we were able to obtain a three-term asymptotic expansion for mentioned characteristics of the process $X(t)$. Note that three-term asymptotic expansion allows us to better analyze the effect of tail distribution on the process.

The rest of this paper is organized as follows. In Section 2 we give mathematical construction of the stochastic process $X(t)$ representing semi-Markovian inventory model of type (s, S) and discuss the ergodicity of the process. In Section 3 we give exact expression and obtain asymptotic expansion for ergodic distribution function of the process $X(t)$ respectively. We then prove the weak convergence theorem for aforementioned distribution function. In Section 4 we propose exact formulas for moments of ergodic distribution of $X(t)$ and obtain asymptotic expansion for these moments.

2. Mathematical construction and ergodicity of the process $X(t)$

Let $\{\eta_n\}_{n \geq 1}$ and $\{\xi_n\}_{n \geq 1}$ be two independent sequences of random variables defined on same probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ such that random variables in each sequence are independent, identically distributed, and positive. Suppose the distributions of random variables ξ_n and η_n are denoted with $\Phi(t)$ and $F(x)$ respectively and defined as:

$$\Phi(t) = \mathbb{P}(\xi_1 \leq t), \ F(x) = \mathbb{P}(\eta_1 \leq x), \ x > 0$$

The renewal sequences $\{T_n\}$ and $\{S_n\}$ are defined as:

$$T_0 = S_0 = 0, \ T_n = \sum_{i=1}^{n} \xi_i, \ S_n = \sum_{i=1}^{n} \eta_i, \ n \geq 1.$$

Now define a sequence of integer valued random variables $\{N_n\}$, $n \geq 0$ as follows:

$$N_0 = 0, \; N_{n+1} = \inf\left\{k \geq N_n + 1 : \; S - (S_k - S_{N_n}) < s\right\}, \; n \geq 1.$$

Moreover, let

$$\tau_0 = 0, \; \tau_n = T_{N_n} = \sum_{i=1}^{N_n} \xi_i, \; n \geq 1, \; \nu(t) = \max\{n \geq 0 : \; T_n \leq t\}, \; t \geq 0.$$

Under these assumptions for $t \in [\tau_n, \tau_{n+1})$ the desired stochastic process $X(t)$ is constructed as follows:

$$X(t) = S - \left(\eta_{N_n+1} + \ldots + \eta_{\nu(t)}\right) = S - \left(S_{\nu(t)} - S_{N_n}\right), \; n \geq 0. \tag{2.1}$$

The main purpose of this article is to investigate the asymptotic behavior of ergodic distribution and moments of ergodic distribution of the process for sufficiently large values of parameter $\beta \equiv S - s$. In this research we studied with $\Gamma(g)$ distributed demand random variables. More specifically we assume that demand random variables have logistic distribution with tail distribution defined as follows:

$$\bar{F}(x) = C/(1 + e^x) \tag{2.2}$$

It has been proven that for the distribution in statement (2.2), $\bar{F} \in \Gamma(1)$ and

$$\bar{F}(x - y)/\bar{F}(x) \to e^y, \; x \to \infty$$

see Mitov and Omey [18] page (77). Ergodicity of the process X(t) is given by Nasirova et al. [19] with the following proposition.

Proposition 2.1 *Let the initial sequences $\{\xi_n\}$ and $\{\eta_n\}$, $n \geq 1$ satisfy the following supplementary conditions:*

1. *$E(\xi_1) < \infty$,*

2. *$0 < E\left(\eta_1^2\right) < \infty$,*

3. *$\{\eta_i\}$, $i \geq 1$ are nonarithmetic random variables,*

Then, the process $X(t)$ is ergodic.

3. Exact expression and asymptotic expansion for ergodic distribution function of the process $X(t)$

Let the conditions of Proposition (2.1) be satisfied. Then the ergodic distribution function of the process $X(t)$ has the following explicit form

$$Q_X(x) \equiv \lim_{t \to \infty} \mathbb{P}\{X(t) \leq x\} = 1 - \frac{U(S - x)}{U(S - s)}, \; s \leq x \leq S \tag{3.1}$$

where $U(x) = \sum_{n=0}^{\infty} F^{*n}(x)$ is renewal function generated by demand random variables η_n. For ease of notation, we introduce the standardized version of the process $X(t)$ as follows:

$$W_\beta(t) \equiv \frac{X(t) - s}{\beta}, \; \beta \equiv S - s$$

From (3.1) ergodic distribution function of the process $W_\beta(x) \equiv \frac{1}{\beta}(X(t) - s)$ is obtained as follows:

$$Q_{W_\beta}(x) = 1 - \frac{U(\beta(1-x))}{U(\beta)}, \ \beta \equiv S - s. \tag{3.2}$$

It is known that exact expressions of the renewal function is not easy to obtain except some special distributions like exponential, Erlang, etc. Using asymptotic tools to explore some of the characteristics of semi-Markovian inventory model is a general approach in this case. Asymptotic expansion problem for renewal function is a long and celebrated history. For a detailed literature review on asymptotic expansion of renewal function generated different classes of distributions see [5, 9–11]. We will use the asymptotic expansion (1.4) provided by Mitov and Omey [18] in order to find asymptotic expansion for ergodic distribution function of the process $Q_{W_\beta}(x)$ given with (3.2).

Theorem 3.1 *Let the conditions of Proposition (2.1) be satisfied. Suppose $F(x)$ is nonsingular Logistic distribution, i.e.:*

$$\bar{F}(x) = \mathbb{P}(\eta_1 > x) = C/(1 + e^x). \tag{3.3}$$

Then for each $x \in (0,1)$ asymptotic expansion for ergodic distribution function of the process $Q_{W_\beta}(x)$ defined with statement (3.2) is obtained as follows:

$$Q_{W_\beta}(x) = \frac{\beta x}{\beta + \mu_e} - \frac{2\beta(1-x)}{\mu_1(\beta + \mu_e)^2(1 + e^\beta)}$$

$$+ \frac{2}{\mu_1(\beta + \mu_e)\left(1 + e^{\beta(1-x)}\right)} + O\left(\frac{1}{\beta^2 e^{\beta(1-x)}}\right), \ \beta \to \infty. \tag{3.4}$$

Here $\mu_i = E(X^i)$ for $i = 1, 2$, $\mu_e = \mu_2/2\mu_1$ is expected value of equilibrium distribution F_e and $x \in (0,1)$ in statement (3.4).

Proof   From (1.3) as $\beta \to \infty$ asymptotic expansion for renewal function generated by logistic distributed random variables obtained as follows:

$$U(\beta) = \frac{\beta}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - \frac{1}{\mu_1^2}\frac{2}{1 + e^\beta}. \tag{3.5}$$

From (3.5) as $\beta \to \infty$ we obtain

$$U(\beta)^{-1} = \frac{\mu_1}{\beta}\left\{\left(1 + \frac{\mu_e}{\beta}\right)\left[1 - \frac{2}{\mu_1\beta\left(1 + e^\beta\right)}\frac{\beta}{\beta + \mu_e}\right]\right\}^{-1}$$

$$= \frac{\mu_1}{\beta}\left(1 + \frac{\mu_e}{\beta}\right)^{-1}\left[1 + \frac{2}{\mu_1(\beta + \mu_e)\left(1 + e^\beta\right)} + O\left(\frac{1}{\beta^2 e^{2\beta}}\right)\right]$$

$$= \frac{\mu_1}{\beta + \mu_e} + \frac{2}{(\beta + \mu_e)^2\left(1 + e^\beta\right)} + O\left(\frac{1}{\beta^3 e^{2\beta}}\right). \tag{3.6}$$

On the other hand for each $x \in (0,1)$ as $\beta \to \infty$ we have

$$U(\beta(1-x)) = \frac{\beta(1-x)}{\mu_1} + \frac{\mu_e}{\mu_1} - \frac{1}{\mu_1^2}\frac{2}{\left(1 + e^{\beta(1-x)}\right)}. \tag{3.7}$$

Using (3.6) and (3.7), for each $x \in (0,1)$ as $\beta \to \infty$ we obtain

$$\frac{U\left(\beta(1-x)\right)}{U\left(\beta\right)} = \left\{\frac{\beta(1-x)}{\mu_1} + \frac{\mu_e}{\mu_1} - \frac{1}{\mu_1^2}\frac{2}{\left(1+e^{\beta(1-x)}\right)}\right\}$$

$$\left\{\frac{\mu_1}{\beta + \mu_e} + \frac{2}{\left(\beta + \mu_e\right)^2\left(1+e^{\beta}\right)} + O\left(\frac{1}{\beta^3 e^{2\beta}}\right)\right\}$$

$$= \frac{\beta - \beta x + \mu_e}{\beta + \mu_e} + \frac{2\beta\left(1-x\right)}{\mu_1\left(\beta + \mu_e\right)^2\left(1+e^{\beta}\right)}$$

$$- \frac{2}{\mu_1(\beta + \mu_e)\left(1+e^{\beta(1-x)}\right)} + O\left(\frac{1}{\beta^2 e^{\beta(1-x)}}\right) \tag{3.8}$$

Taking into account (3.2) we obtain the following result as $\beta \to \infty$

$$Q_{W_\beta}\left(x\right) = 1 - \frac{U(\beta(1-x))}{U(\beta)}$$

$$= \frac{\beta x}{\beta + \mu_e} - \frac{2\beta(1-x)}{\mu_1(\beta + \mu_e)^2(1+e^{\beta})}$$

$$+ \frac{2}{\mu_1(\beta + \mu_e)(1+e^{\beta(1-x)})} + O(\frac{1}{\beta^2 e^{\beta(1-x)}}). \tag{3.9}$$

$\square$

We should note that, since $\bar{F}(0) = 1$ then we can take $C = 2$ in (3.3). Corollary 3.2 is a straightforward result of Theorem 3.1.

Corollary 3.2 *(Weak Convergence) Let the conditions of Theorem 3.1 be satisfied. Then for $\beta \equiv S - s \to \infty$ and for each $x \in (0,1)$ ergodic distribution of the process $W_\beta\left(t\right)$ converges weakly to uniform distribution, defined in $[0,1]$. That means*

$$Q_{W_\beta}\left(x\right) \to G(x) = x, \; \beta \to \infty.$$

Proof Since $\mu_2 < \infty$, then $\mu_e = \mu_2/2\mu_1 < \infty$. Using L'Hospital's rule it is easy to see that for $x \in (0,1)$

$$lim_{\beta\to\infty}\frac{2\beta(1-x)}{\mu_1(\beta + \mu_e)^2(1+e^{\beta})} = 0,$$

Moreover, using basic properties of limit we obtain following straightforward results for $x \in (0,1)$:

$$\lim_{\beta\to\infty}\frac{2}{\mu_1(\beta + \mu_e)(1+e^{\beta(1-x)})} = 0, \; \lim_{\beta\to\infty}\frac{1}{\beta^2 e^{\beta(1-x)}} = 0, \; \lim_{\beta\to\infty}\frac{\beta x}{\beta + \mu_e} = x$$

Hence,

$$Q_{W_\beta}\left(x\right) \to x, \; \beta \to \infty.$$

$\square$

Note that since $W_\beta\left(t\right) = \left(X\left(t\right) - s\right)/\beta$ then $X\left(t\right) = s + \beta W_\beta\left(t\right)$. From Corollary 3.2 we can observe that under the conditions of Theorem 3.1 ergodic distribution of the process $X\left(t\right)$ is close to uniform distribution defined in $[s, S]$ for sufficiently large values of β. That means

$$Q_X(x) \approx \frac{X - s}{S - s}, \; x \in [s, S]$$

4. Exact expression and asymptotic expansion for moments of ergodic distribution of the process $X(t)$

In this section we will investigate asymptotic behavior of the moments of ergodic distribution of the process $X(t)$ for a sufficiently large values of $\beta = S - s$. Before obtaining asymptotic results let us introduce the exact formula first. Note that for simplicity of the notations n^{th} order moments of the ergodic distribution of the process $X(t)$ are denoted by $E(X^n) = lim_{t\to\infty}E(X^n(t))$, $n = 1, 2, 3, \dots$. In addition, we put

$$\tilde{X}(t) \equiv X(t) - s;\ E\left(\tilde{X}^n\right) = lim_{t\to\infty}E\left(\tilde{X}^n(t)\right)\ n = 1, 2, 3, \dots.$$

Taking into account these notations, the following proposition can be stated.

Proposition 4.1 *Let the conditions of the Proposition 2.1 be satisfied. If n^{th} order moments of ergodic distribution of the process $\tilde{X}(t)$ exists and is finite, then it can be represented as follows using renewal function $U(x)$.*

$$E\left(\tilde{X}^n\right) = \frac{nU_n(\beta)}{U(\beta)}, \tag{4.1}$$

where

$$U_n(\beta) \equiv \beta^{n-1} * U(\beta) \equiv \int_0^\beta (\beta - t)^{n-1} U(t)\, dt,\ n \geq 1. \tag{4.2}$$

Note that $U(x)$ is renewal function of the demand random variables $\{\eta_i\}_{i\geq 1}$.

See for example Khaniyev et al. [13] for the proof of Proposition 4.1. In order to obtain asymptotic expansion for the moments of ergodic distribution of the process $\tilde{X}(t) \equiv X(t) - s$ we will use intuitive approximation (1.3) provided by Mitov and Omey [18]. In order to reach the asymptotic expansions, we must first prove the following lemmas.

Lemma 4.2 *For any $x > 0$ and $n \geq 1$ let us define*

$$G_n(x) = h_1(x) * h_2(x) = \int_0^x (x - t)^{n-1} e^{-t} dt, \tag{4.3}$$

where

$$h_1(x) = x^{n-1},\ h_2(x) = e^{-x}.$$

Then

$$G_n(x) = \sum_{r=0}^{n-1} (-1)^r \frac{x^{n-r-1}}{(n-r-1)!} + (-1)^n e^{-x}. \tag{4.4}$$

Proof Let us take Laplace transform of $G_n(x)$ as a first step.

$$\tilde{G}_n(\lambda) = \tilde{h}_1(\lambda)\tilde{h}_2(\lambda)$$

, where $\tilde{G}_n(\lambda)$, $\tilde{h}_1(\lambda)$, and $\tilde{h}_2(\lambda)$ are Laplace transforms of $G_n(x)$, $h_1(x)$ and $h_2(x)$ respectively. Hence,

$$\tilde{h}_1(\lambda) = \int_0^\infty x^{n-1} e^{-\lambda x} dx = \frac{\Gamma(n)}{\lambda^n} = \frac{(n-1)!}{\lambda^n}$$

and

$$\tilde{h}_2(\lambda) = \int_0^\infty e^{-x}e^{-\lambda x}dx = \frac{\Gamma(1)}{\lambda+1} = \frac{1}{\lambda+1}.$$

From here $\tilde{G}_n(\lambda)$ is obtained as follows:

$$\tilde{G}_n(\lambda) = \frac{(n-1)!}{\lambda^n(\lambda+1)} = (n-1)!\left[\frac{1}{\lambda^n} - \frac{1}{\lambda^{n-1}} + \frac{1}{\lambda^{n-2}} - \cdots\right]$$

$$= (n-1)!\sum_{r=0}^{n-1}(-1)^r\frac{1}{\lambda^{n-r}} + (-1)^n\frac{1}{\lambda+1} \tag{4.5}$$

$\square$

Taking inverse Laplace transform of (4.5) we obtain

$$G_n(x) = \sum_{r=0}^{n-1}(-1)^r\frac{x^{n-r-1}}{(n-r-1)!} + (-1)^ne^{-x}.$$

as desired.

Lemma 4.3 *For any $n \geq 1$ let us define*

$$J(\beta) = \int_0^\beta (\beta-t)^{n-1}\frac{1}{1+e^t}dt, \tag{4.6}$$

then we get

$$J(\beta) = \left[\sum_{r=0}^{n-1}\frac{(-1)^r a_r}{(n-r-1)!}\beta^{n-r-1} + O\left(e^{-\beta}\right)\right]; \tag{4.7}$$

where

$$a_r = \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k^{r+1}}.$$

Proof Using Taylor expansion of $1/\left(1+e^{-\beta(1-x)}\right)$ we obtain

$$J(\beta) = \int_0^\beta (\beta-t)^{n-1}\frac{1}{1+e^t}dt = \beta^n\int_0^1 \frac{x^{n-1}}{e^{\beta(1-x)}\left(1+e^{-\beta(1-x)}\right)}dx$$

$$= \beta^n\sum_{k=1}^\infty (-1)^{k-1}\underbrace{\int_0^1 x^{(n-1)}e^{-\beta k(1-x)}dx}_{I_1(k\beta)} \tag{4.8}$$

Let us define $I_1(\beta)$ as follows:

$$I_1(\beta) = \int_0^1 x^{n-1}e^{-\beta(1-x)}dx$$

$$= \frac{1}{\beta^n}\int_0^\beta (\beta-t)^{n-1}e^{-t}dt = \frac{1}{\beta^n}G_n(\beta). \tag{4.9}$$

Using statement (4.4) on the right hand side of (4.9) we obtain

$$I_1(\beta) = \frac{1}{\beta^n}\left[\sum_{r=0}^{n-1}(-1)^r\frac{\beta^{n-r-1}}{(n-r-1)!} + (-1)^n e^{-\beta}\right].$$

Similarly,

$$I_1(k\beta) = \frac{1}{(k\beta)^n}\left[\sum_{r=0}^{n-1}(-1)^r\frac{(k\beta)^{n-r-1}}{(n-r-1)!} + (-1)^n e^{-k\beta}\right].$$

Hence,

$$\sum_{k=1}^{\infty}(-1)^{k-1}I_1(k\beta) = \sum_{k=1}^{\infty}(-1)^{k-1}\sum_{r=0}^{n-1}\frac{(-1)^r}{(n-r-1)!(k\beta)^{r+1}} + \sum_{k=1}^{\infty}(-1)^{k+n-1}(k\beta)^{-n}e^{-k\beta}$$

$$\approx \sum_{r=0}^{n-1}\frac{(-1)^r}{(n-r-1)!\beta^{r+1}}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^{r+1}} + O\left(\beta^{-n}e^{-\beta}\right)$$

$$= \sum_{r=0}^{\infty}\frac{a_r(-1)^r}{(n-r-1)!\beta^{r+1}} + O\left(\beta^{-n}e^{-\beta}\right) \tag{4.10}$$

Desired result holds taking into account

$$J(\beta) = \int_0^{\beta}(\beta - t)^{n-1}\frac{1}{1+e^t} = \beta^n\sum_{k=1}^{\infty}(-1)^{k-1}I_1(k\beta).$$

$\square$

Lemma 4.4 *Let the conditions of the PropositionS 2.1 and 4.1 be satisfied. Then as $\beta \to \infty$ following asymptotic expansion holds for $U_n(\beta)$ defined with (4.2).*

$$U_n(\beta) = \frac{1}{\mu_1 n(n+1)}\beta^{n+1} + \frac{\mu_2}{2\mu_1^2 n}\beta^n - \frac{2a_0}{\mu_1^2(n-1)!}\beta^{n-1} + O\left(\beta^{n-2}\right), \tag{4.11}$$

where $\mu_i = E(\eta_1^i)$ for $i = 1,2$, $n \geq 1$ and $a_r = \sum_{k=1}^{\infty}(-1)^{k-1}/k^{r+1}$. Note that by alternating series test the series $a_r = \sum_{k=1}^{\infty}(-1)^{k-1}/k^{r+1}$ is convergent for all $r \geq 0$.

Proof Using the results of LemmaS 4.2 and 4.3 on the right hand side of (4.2) we obtain the following asymptotic expansion:

$$U_n(\beta) = \beta^{n-1} * U(\beta) = \int_0^{\beta}(\beta - t)^{n-1}U(t)dt$$

$$= \frac{1}{\mu_1}\int_0^{\beta}(\beta - t)^{n-1}tdt + \frac{\mu_2}{2\mu_1^2}\int_0^{\beta}(\beta - t)^{n-1}dt - \frac{2}{\mu_1^2}\int_0^{\beta}(\beta - t)^{n-1}\frac{1}{1+e^t}dt$$

$$= \frac{\beta^{n+1}}{\mu_1 n(n+1)} + \frac{\mu_2}{2\mu_1^2 n}\beta^n - \frac{2}{\mu_1^2}\sum_{r=0}^{n-1}\frac{(-1)^r a_r\beta^{n-r-1}}{(n-r-1)!} + O\left(e^{-\beta}\right)$$

$$= \frac{\beta^{n+1}}{\mu_1 n(n+1)} + \frac{\mu_2}{2\mu_1^2 n}\beta^n - \frac{2a_0}{\mu_1^2(n-1)!}\beta^{n-1} + O\left(\beta^{n-2}\right) + O\left(e^{-\beta}\right) \tag{4.12}$$

Statement (4.11) holds, comparing last two terms of (4.12). $\qquad\square$

Theorem 4.5 *Let the conditions of the Propositions 2.1 and 4.1 be satisfied. Then n^{th} order moments of ergodic distribution of the process $\tilde{X}(t) = X(t) - s$ satisfies the following asymptotic expansion as $\beta \to \infty$.*

$$E\left(\tilde{X}^n(\beta)\right) = \frac{1}{(n+1)(\beta+\mu_e)}\beta^{n+1} + \frac{\mu_e}{(\beta+\mu_e)}\beta^n - \frac{2na_0}{\mu_1(n-1)!(\beta+\mu_e)}\beta^{n-1} + O\left(\frac{\beta^{n-2}}{\beta+\mu_e}\right), \qquad (4.13)$$

where

$$a_0 = \sum_{k=1}^{\infty}(-1)^{k-1}/k = \ln 2, \ \mu_1 = 2\ln 2, \ \mu_2 = \frac{\pi^2}{3}, \ \mu_e = \frac{\mu_2}{2\mu_1} = \frac{\pi^2}{12\ln 2}.$$

Proof   Using (3.6), (4.11), and basic rules of asymptotic theory for any $n \geq 1$ we obtain

$$E\left(\tilde{X}^n(\beta)\right) = \frac{nU_n(\beta)}{U(\beta)}$$

$$= \left\{\frac{1}{\mu_1(n+1)}\beta^{n+1} + \frac{\mu_2}{2\mu_1^2}\beta^n - \frac{2a_0 n}{\mu_1^2(n-1)!}\beta^{n-1} + O\left(\beta^{n-2}\right)\right\}$$

$$\left\{\frac{\mu_1}{\beta+\mu_e} + \frac{2}{(\beta+\mu_e)^2(1+e^\beta)} + O\left(\frac{1}{\beta^3 e^{2\beta}}\right)\right\}$$

$$= \frac{1}{(n+1)(\beta+\mu_e)}\beta^{n+1} + \frac{\mu_e}{(\beta+\mu_e)}\beta^n$$

$$- \frac{2na_0}{\mu_1(n-1)!(\beta+\mu_e)}\beta^{n-1} + O\left(\frac{\beta^{n-2}}{\beta+\mu_e}\right), \ \beta \to \infty.$$

$\qquad\square$

Corollary 4.6 *Let the conditions of Propositions 2.1 and 4.1 be satisfied. Then, the asymptotic expansion for the first and second initial moments of the ergodic distribution of the process $\tilde{X}(t) = X(t) - s$ can be written as follows when $\beta \to \infty$*

$$E(\tilde{X}(\beta)) = \frac{1}{2(\beta+\mu_e)}\beta^2 + \frac{\mu_e}{(\beta+\mu_e)}\beta - \frac{2a_0}{\mu_1(\beta+\mu_e)} + O\left(\frac{1}{\beta(\beta+\mu_e)}\right) \qquad (4.14)$$

$$E(\tilde{X}^2(\beta)) = \frac{1}{3(\beta+\mu_e)}\beta^3 + \frac{\mu_e}{(\beta+\mu_e)}\beta^2 - \frac{4a_0}{\mu_1(\beta+\mu_e)}\beta + O\left(\frac{1}{(\beta+\mu_e)}\right). \qquad (4.15)$$

5. Conclusion

This study is intended to investigate a classical semi-Markovian inventory model of type (s, S) with $\Gamma(g)$ distributed demand random variables. $\Gamma(g)$ is a special class which contains examples such as Weibull, logistic, and extreme value distributions that are frequently encountered in extreme value theory. The results of this work are based on the study by Mitov and Omey [18] where they examined lots of specific features of the class of $\Gamma(g)$. Moreover, they provide an approximation for renewal function for this class and their approximation

cover many of the known results in the literature. Using asymptotic results presented by Mitov and Omey [18] we obtained three terms of asymptotic expansion for ergodic distribution and moments of ergodic distribution of a stochastic process $X(t)$ representing inventory model of type (s, S).

A semi-Markovian inventory model of type (s, S) has been comprehensively studied in the literature with heavy tailed demand random variables but only two-term results were achieved previously. The three-term expansions obtained in this study have enabled us to analyze the effect of the tail distribution on the process better comparing with preceding studies.

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