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Further remarks on group-2-groupoids

Sedat Temel

Department of Mathematics, Recep Tayyip Erdogan University, Turkey (sedat.temel@erdogan.edu.tr)

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Abstract

The aim of this paper is to obtain a group-2-groupoid as a 2-groupoid object in the category of groups and also as a special kind of an internal category in the category of group-groupoids. Corresponding group-2-groupoids, we obtain some categorical structures related to crossed modules and group-groupoids and prove categorical equivalences between them. These results enable us to obtain 2-dimensional notions of group-groupoids.

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Keywords: *crossed module; group-groupoid; 2-groupoid.*

1. Introduction

There are several 2-dimensional notions of groupoids such as double groupoids, 2-groupoids, and crossed modules over groupoids. The purpose of this paper is to obtain 2-dimensional notions of group-groupoids which are internal groupoids in the category of groups and widely used under the name of 2-groups.

The term "categorification", which was first used by Louis Crane [13] in the context of mathematical physics, is the process of replacing set-theoretic theorems by category-theoretic concepts. The aim of categorification is to develop a richer case of existing mathematics by replacing sets with categories, functions with functors and equations between functions with natural isomorphisms between functors. In this approach, the categorified version of a group is called a group-groupoid [2, 5]. Group-groupoids, which are also known as $\mathcal{G}$-groupoids [6] or 2-groups [4], are internal categories (hence internal groupoids) in the

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category **Gp** of groups [22, 23]. Equivalently, group-groupoids can be thought as group objects in the category **Cat** of small categories [6, 23].

Another useful viewpoint of group-groupoids is to think them as crossed modules over groups. Crossed modules which can be viewed as 2-dimensional groups [7] are widely used in homotopy theory [8], homological algebra [16], and algebraic K-theory [21]. The well-known categorical equivalence between crossed modules and group-groupoids is proved by Brown and Spencer [6]. This equivalence is introduced in [4] by obtaining a group-groupoid as a 2-category with a unique object. Crossed modules, and their higher dimensional analogues, provide algebraic models for homotopy n-types; the group-2-groupoids of this paper in principle provide algebraic models for certain homotopy 3-types.

In the previous paper [1], the notions of a group-2-groupoid were introduced and compared with a corresponding structure related to crossed modules over groups. On the other hand, the main objective of this paper is to obtain the structure of a group-2-groupoid as a 2-groupoid object in the category of groups and also as a special kind of internal category in the category of group-groupoids. In section 4, we present the notion of crossed modules over group-groupoids and prove that there is a categorical equivalence between group-2-groupoids and crossed modules over group-groupoids using the categorical equivalence between 2-groupoids and crossed modules over groupoids given in [17]. In section 5, we show that group-2-groupoids are categorically equivalent to special kind of internal categories in the category of crossed modules.

2. Preliminaries

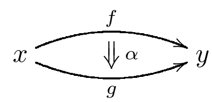
Let $\mathcal{C}$ be a finitely complete category and D_0, D_1 are objects of the ambient category $\mathcal{C}$. An *internal category* $\mathcal{D} = (D_0, D_1, s, t, \varepsilon, m)$ in $\mathcal{C}$ consists of an object D_0 in $\mathcal{C}$ called the object of objects and an object D_1 in $\mathcal{C}$ called the object of arrows (i.e. morphisms), together with morphisms $s, t\colon D_1 \to D_0$, $\varepsilon\colon D_0 \to D_1$ in $\mathcal{C}$ called the source, the target and the identity maps, respectively,

$$D_1 \underset{t}{\overset{s}{\rightrightarrows}} D_0 \quad (\varepsilon\colon D_0 \to D_1)$$

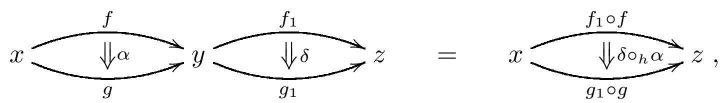
such that $s\varepsilon = t\varepsilon = 1_{D_0}$ and a morphism $m\colon D_1 \times_{D_0} D_1 \to D_1$ of $\mathcal{C}$ called the composition map (usually expressed as $m(f, g) = g \circ f$) where $D_1 \times_{D_0} D_1$ is the pullback of s, t such that $\varepsilon s(f) \circ f = f = f \circ \varepsilon s(f)$ [22]. An *internal groupoid* in $\mathcal{C}$ is an internal category with a morphism $\eta\colon D_1 \to D_1$, $\eta(f) = \overline{f}$ in $\mathcal{C}$ called inverse such that $\overline{f} \circ f = 1_{s(f)}$, $f \circ \overline{f} = 1_{t(f)}$.

We write $C(x, y)$ for all morphisms from x to y where $x, y \in C_0$. If $C(x, y) = \varnothing$ for all $x, y \in C_0$ such that $x \neq y$, then $\mathcal{C}$ is called totally disconnected category.

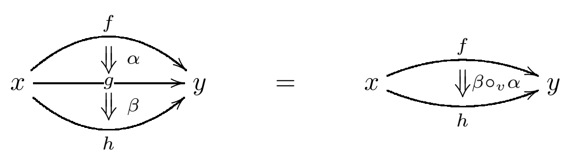
We introduce the definition of a 2-category as given in [4]. A *2-category* $\mathcal{C} = (C_0, C_1, C_2)$ consists of a set of objects C_0, a set of 1-morphisms C_1, and a set of 2-morphisms C_2 as follows:



with maps $s\colon C_1 \to C_0,\ s(f) = x,\ s_h\colon C_2 \to C_0,\ s_h(\alpha) = x,\ s_v\colon C_2 \to C_1,\ s_v(\alpha) = f,\ t\colon C_1 \to C_0,\ t(f) = y,\ t_h\colon C_2 \to C_0,\ t_h(\alpha) = y,\ t_v\colon C_2 \to C_1,\ t_v(\alpha) = g$, called the source and the target maps, respectively, the composition of 1-morphisms as in an ordinary category, the associative horizontal composition of 2-morphisms $\circ_h\colon C_2 \times_{C_0} C_2 \to C_2$ as



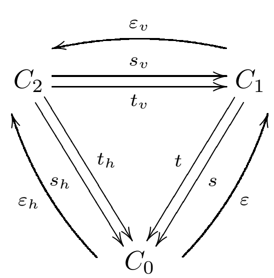
where $C_2 \times_{C_0} C_2 = \{(\alpha,\delta) \in C_2 \times C_2 | s_h(\delta) = t_h(\alpha)\}$ and the associative vertical composition of 2-morphisms $\circ_v\colon C_2 \times_{C_1} C_2 \to C_2$ as



where $C_2 \times_{C_1} C_2 = \{(\alpha,\beta) \in C_2 \times C_2 | s_v(\beta) = t_v(\alpha)\}$ such that satisfying the following interchange rule:

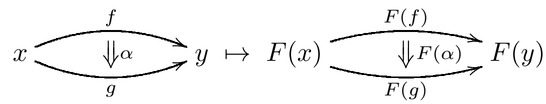
$$(\theta \circ_v \delta) \circ_h (\beta \circ_v \alpha) = (\theta \circ_h \beta) \circ_v (\delta \circ_h \alpha)$$

whenever one side makes sense, and the identity maps $\varepsilon\colon C_0 \to C_1, \varepsilon(x) = 1_x$, $\varepsilon_h\colon C_0 \to C_2$, $\varepsilon_h(x) = 1_{1_x}$ such that $\alpha \circ_h 1_{1_x} = \alpha = 1_{1_y} \circ_h \alpha$ and $\varepsilon_v\colon C_1 \to C_2$, $\varepsilon_v(f) = 1_f$ such that $\alpha \circ_v 1_f = \alpha = 1_g \circ_v \alpha$. Therefore, the construction of a 2-category $\mathcal{C} = (C_0, C_1, C_2)$ contains compatible category structures $\mathcal{C}_1 = (C_0, C_1, s, t, \varepsilon, \circ)$, $\mathcal{C}_2 = (C_0, C_2, s_h, t_h, \varepsilon_h, \circ_h)$, and $\mathcal{C}_3 = (C_1, C_2, s_v, t_v, \varepsilon_v, \circ_v)$ such that the following diagram commutes.



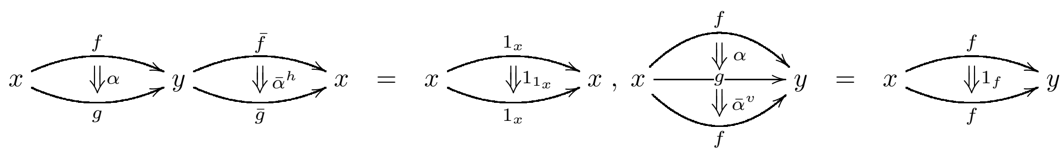
Let $\mathcal{C}$ and $\mathcal{C}'$ be 2-categories. A *2-functor* is a map $F\colon \mathcal{C} \to \mathcal{C}'$ sending each object of $\mathcal{C}$ to an object of $\mathcal{C}'$, each 1-morphism of $\mathcal{C}$ to 1-morphism of $\mathcal{C}'$ and

2-morphism of $\mathcal{C}$ to 2-morphism of $\mathcal{C}'$ as follows:



such that $F(f_1 \circ f) = F(f_1) \circ F(f)$, $F(\delta \circ_h \alpha) = F(\delta) \circ_h F(\alpha)$, $F(\beta \circ_v \alpha) = F(\beta) \circ_v F(\alpha)$, $F(1_{1_x}) = 1_{F(1_x)} = 1_{1_{F(x)}}$, $F(1_f) = 1_{F(f)}$. Hence 2-categories form a category which is denoted by **2Cat** [24].

A strict 2-groupoid is a 2-category all of whose 1-morphisms are invertible and in which all 2-morphisms are invertible horizontally and vertically.



Let $\mathcal{G}, \mathcal{G}'$ be 2-groupoids. A *morphism of 2-groupoids* is a 2-functor $F\colon \mathcal{G} \to \mathcal{G}'$ which preserves the 2-groupoid structures. Thus, 2-groupoids and their morphisms form a category which is denoted by **2Gpd** [24].

A *group-groupoid* is an internal category in **Gp** [22]. Also, a group-groupoid can be obtained as a group object in the category **Cat** of small categories (or in **Gpd**). A *morphism of group-groupoids* is a morphism of groupoids which preserves group structures. Hence we can define the category of group-groupoids, which is denoted by **2Gp** or **GpGd**. For further details about group-groupoids, see [24, 6, 4].

By a crossed module as defined by Whitehead, it is meant a pair M, N of groups together with an action $\bullet\colon N \times M \to M$ of groups and a morphism $\partial\colon M \to N$ of groups such that $\partial(n \bullet m) = n\partial(m)n^{-1}$ and $\partial(m) \bullet m' = mm'm^{-1}$ [28, 29].

Let $K = (M, N, \partial, \bullet)$, $K' = (M', N', \partial', \bullet')$ be crossed modules and $\lambda_1\colon N \to N'$, $\lambda_2\colon M \to M'$ be morphisms of groups. If λ_1, λ_2 satisfies the conditions $\lambda_1\partial = \partial'\lambda_2$ and $\lambda_2(n \bullet m) = \lambda_1(n) \bullet' \lambda_2(m)$, then $\langle \lambda_2, \lambda_1 \rangle\colon K \to K'$ is called *morphism of crossed modules* [6]. Hence crossed modules and their morphisms form a category which we denote by **Cm**.

The following theorem was proved by Brown and Spencer in [6]:

Theorem 2.1. *The category of group-groupoids and the category of crossed modules are equivalent.*

Let $\mathcal{G} = (X, G)$ and $\mathcal{H} = (X, H)$ be groupoids over the same object set X such that $\mathcal{H}$ is totally disconnected. We recall from [8, 17, 11] that an action of $\mathcal{G}$ on $\mathcal{H}$ is a partially defined map

$$\bullet\colon G\times H\to H,\quad (g,h)\mapsto g\bullet h$$

such that the following conditions satisfies

[AG 1] $g\bullet h$ is defined iff $t(h)=s(g)$, and $t(g\bullet h)=t(g)$,
[AG 2] $(g_2\circ g_1)\bullet h=g_2\bullet(g_1\bullet h)$,
[AG 3] $g\bullet(h_2\circ h_1)=(g\bullet h_2)\circ(g\bullet h_1)$, for $h_1,h_2\in H(x,x)$ and $g\in G(x,y)$,
[AG 4] $1_x\bullet h=h$, for $h\in H(x,x)$.

From this conditions, it can be easily obtain that $g\bullet 1_x=1_y$, for $g\in G(x,y)$.

Using this action of $\mathcal{G}$ on $\mathcal{H}$, we can obtain a groupoid which is called semi-direct product of $\mathcal{G}$ and $\mathcal{H}$ denoted by $G\ltimes H$. Let $x\xrightarrow{g}y\xrightarrow{h}y$ are morphisms of $\mathcal{G}$ and $\mathcal{H}$, respectively, then (g,h) is a morphism as follows

$$x\xrightarrow{(g,h)}y$$

where the structure maps are defined by $s(g,h)=s(g)$, $t(g,h)=t(g)$, $\varepsilon(x)=(1_x,1_x)$. If

$$x\xrightarrow{g}y\xrightarrow{h}y\xrightarrow{g_1}z\xrightarrow{h_1}z$$

then the composition of morphisms is defined by

$$(g_1,h_1)\circ(g,h)=(g_1\circ g,h_1\circ(g_1\bullet h)).$$

The notion of crossed modules over groupoids is introduced by Brown-Higgins [9, 10] and Brown-Icen [11]. Let $\mathcal{G}=(X,G)$ and $\mathcal{H}=(X,H)$ be groupoids over the same object set X such that $\mathcal{H}$ is totally disconnected. A crossed module $\mathcal{K}=(\mathcal{H},\mathcal{G},\partial,\bullet)$ over groupoids consists of a morphism $\partial=(1,\partial)\colon\mathcal{H}\to\mathcal{G}$ of groupoids which is identity on objects together with an action $\bullet\colon G\times H\to H$ of groupoids which satisfies $\partial(g\bullet h)=g\circ\partial(h)\circ\overline{g}$ and $\partial(h)\bullet h_1=h\circ h_1\circ\overline{h}$, for $h,h_1\in H(x,x)$ and $g\in G(x,y)$.

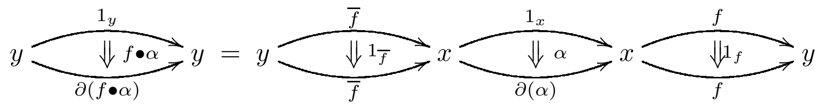
Let $\mathcal{K}=(\mathcal{H},\mathcal{G},\partial,\bullet)$ and $\mathcal{K}'=(\mathcal{H}',\mathcal{G}',\partial',\bullet')$ be crossed modules over groupoids. A *morphism of crossed modules over groupoids* is a mapping $\lambda=\langle\lambda_2,\lambda_1,\lambda_0\rangle\colon\mathcal{K}\to\mathcal{K}'$ which satisfies $\lambda_2\partial=\partial'\lambda_1$ and $\lambda_1(g\bullet h)=\lambda_2(g)\bullet'\lambda_1(h)$ where $(\lambda_0,\lambda_1)\colon\mathcal{H}\to\mathcal{H}'$ and $(\lambda_0,\lambda_2)\colon\mathcal{G}\to\mathcal{G}'$ are morphisms of groupoids. Hence the category of crossed modules over groupoids can be defined which we denoted by **Cmg**.

The following result was proved by Icen in [17]. Since we need some details in section 4, we give a sketch proof in terms of our notations.

Theorem 2.2. *The categories of 2-groupoids and of crossed module over groupoids are equivalent.*

Proof. For any 2-groupoid $\mathcal{G}=(G_0,G_1,G_2)$, we know that $\mathcal{B}=(G_0,G_1)$ is a groupoid. Let $A(x)=\{\alpha\in G_2|s_v(\alpha)=\varepsilon(x)\}$, for $x\in G_0$ and $A=\{A(x)\}_{x\in G_0}$. Then $\mathcal{A}=(G_0,A)$ is a totally disconnected groupoid. Now we define a functor

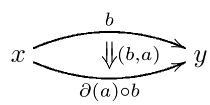
$\gamma\colon \mathbf{2Gpd} \to \mathbf{Cmg}$ as an equivalence of categories such that $\gamma(\mathcal{G}) = (\mathcal{A}, \mathcal{B}, \partial)$ is a crossed module over groupoids with $\partial\colon \mathcal{A} \to \mathcal{B}$, $\partial(\alpha) = t_v(\alpha)$ and an action of groupoids such that $f \bullet \alpha = 1_f \circ_h \alpha \circ_h 1_{\overline{f}}$.



Clearly $\partial(f \bullet \alpha) = f \circ \partial(\alpha) \circ \overline{f}$ and $\partial(\alpha) \bullet \alpha_1 = \alpha \circ_h \alpha_1 \circ_h \overline{\alpha}^h$, for $f \in G_1(x, y)$ and $\alpha, \alpha_1 \in A(x)$.

Let $F = (F_0, F_1, F_2)$ be a morphism of 2-groupoids. Then $\gamma(F) = \langle F_2\big|_A, F_1, F_0\rangle$ is a morphism of crossed modules over groupoids.

Now we define a functor $\theta\colon \mathbf{Cmg} \to \mathbf{2Gpd}$ which is an equivalence of categories. Let $\mathcal{K} = (\mathcal{A}, \mathcal{B}, \partial)$ be a crossed module over groupoids $\mathcal{A} = (X, A)$ and $\mathcal{B} = (X, B)$. Then 2-groupoid $\theta(\mathcal{K}) = (X, B, B \ltimes A)$ is a 2-groupoid which is constructed as in the following way. The set of 2-morphisms is the semi-direct product $B \ltimes A = \{(b, a) | b \in B, a \in A, s(a) = t(a) = t(b)\}$. If $x \xrightarrow{b} y \xrightarrow{a} y$, then (b, a) is a 2-morphism as follows:



where the horizontal composition of 2-morphisms is defined by

$$(b_1, a_1) \circ_h (b, a) = (b_1 \circ b, a_1 \circ (b_1 \bullet a))$$

when $y \xrightarrow{b_1} z \xrightarrow{a_1} z$ and the vertical composition of 2-morphisms is defined by

$$\Big(\partial(a) \circ b, a_2\Big) \circ_v (b, a) = (b, a_2 \circ a)$$

when $y \xrightarrow{a_2} y$. The source and the target maps are defined by $s_h(b, a) = s(b), s_v(b, a) = b, t_h(b, a) = t(b), t_v(b, a) = \partial(a) \circ b$, respectively, the identity maps are defined by $\varepsilon_h(x) = (1_x, 1_x), \varepsilon_v(b) = (b, 1_y)$, and the inversion maps are defined by $\overline{(b, a)}^v = (\partial(a) \circ b, \overline{a}), \overline{(b, a)}^h = (\overline{b}, \overline{b} \bullet \overline{a})$.

Let $\lambda = \langle \lambda_2, \lambda_1, \lambda_0 \rangle$ be a morphism of crossed modules over groupoids. Then

$$\theta(\lambda) = (\lambda_0, \lambda_2, \lambda_2 \times \lambda_1)$$

is a morphism of 2-groupoids.

A natural equivalence $S\colon \theta\gamma \to \mathbf{1_{2Gpd}}$ is defined via the map $S_{\mathcal{G}}\colon \theta\gamma(\mathcal{G}) \to \mathcal{G}$ which is defined to be identity on objects and on 1-morphisms, on 2-morphisms is defined by $\alpha \mapsto (f, \alpha \circ_h 1_{\overline{f}})$. Clearly $S_{\mathcal{G}}$ is an isomorphism and preserves compositions.

Now, given a crossed module $\mathcal{K} = (\mathcal{A}, \mathcal{B}, \partial, \bullet)$ over groupoids, we define a natural equivalence $T\colon \mathbf{1_{Cmg}} \to \gamma\theta$ by a map $T_{\mathcal{K}}\colon \mathcal{K} \to \gamma\theta(\mathcal{K})$ which is defined to be identity on objects and on B, while on A is defined by $a \mapsto (s(a), a)$. $\square$

3. Group-2-groupoids

In [1], a group-2-groupoid is defined as a group object in **2Cat** using similar methods given in [6, 23]. In other words, a group-2-groupoid $\mathcal{G}$ is a small 2-groupoid equipped with the following 2-functors satisfying group axioms, written out as commutative diagrams

(1) $\mu\colon \mathcal{G}\times\mathcal{G}\to\mathcal{G}$ called product,
(2) $inv\colon \mathcal{G}\to\mathcal{G}$ called inverse and
(3) $id\colon \{*\}\to\mathcal{G}$ (where $\{*\}$ is a singleton) called unit or identity.

Then, the product of $x \underset{b}{\overset{a}{\Downarrow\alpha}} y$ and $x' \underset{b'}{\overset{a'}{\Downarrow\alpha'}} y'$ is written by

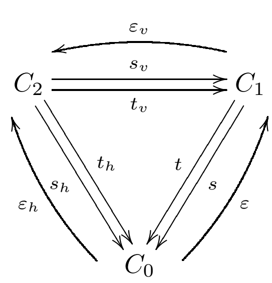
$x\cdot x' \underset{b\cdot b'}{\overset{a\cdot a'}{\Downarrow\alpha\cdot\alpha'}} y\cdot y'$, the inverse of $x \underset{b}{\overset{a}{\Downarrow\alpha}} y$ is $x^{-1} \underset{b^{-1}}{\overset{a^{-1}}{\Downarrow\alpha^{-1}}} y^{-1}$

where $id\{*\} = e \underset{1_e}{\overset{1_e}{\Downarrow 1_{1_e}}} e$. The condition 1 above gives us the following interchange rules

$$(a_1\circ a)\cdot(a_1'\circ a') = (a_1\cdot a_1')\circ(a\cdot a'),$$
$$(\delta\circ_h\alpha)\cdot(\delta'\circ_h\alpha') = (\delta\cdot\delta')\circ_h(\alpha\alpha'),$$
$$(\beta\circ_v\alpha)\cdot(\beta'\circ_v\alpha') = (\beta\cdot\beta')\circ_v(\alpha\cdot\alpha')$$

whenever compositions are defined. We can obtain from the condition 2 that $(a_1\circ a)^{-1} = a_1^{-1}\circ a^{-1}$, $(\delta\circ_h\alpha)^{-1} = \delta^{-1}\circ_h\alpha^{-1}$, $(\beta\circ_v\alpha)^{-1} = \beta^{-1}\circ_v\alpha^{-1}$, $1_x^{-1} = 1_{x^{-1}}$, $1_{1_x}^{-1} = 1_{1_{x^{-1}}}$ and $1_a^{-1} = 1_{a^{-1}}$. Moreover, the structure of a group-2-groupoid $\mathcal{G} = (G_0, G_1, G_2)$ contains compatible group-groupoids $G = (G_0, G_1)$, $G' = (G_0, G_2)$ and $G'' = (G_1, G_2)$ [1].

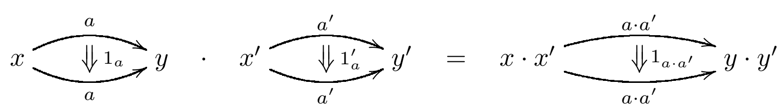
Equivalently we shall describe a group-2-groupoid as a 2-groupoid object in the category **Gp** of groups. Let C_0, C_1 and C_2 be objects of a finitely complete category $\mathcal{C}$. If $\mathcal{C}_1 = (C_0, C_1, s, t, \varepsilon, \circ)$, $\mathcal{C}_2 = (C_0, C_2, s_h, t_h, \varepsilon_h, \circ_h)$, and $\mathcal{C}_3 = (C_1, C_2, s_v, t_v, \varepsilon_v, \circ_v)$ are internal categories in $\mathcal{C}$ such that the following diagram commutes whenever the usual interchange rule satisfies between $\circ_h$ and $\circ_v$, then (C_0, C_1, C_2) is called an internal 2-category in $\mathcal{C}$.



Proposition 3.1. *A 2-category object in* **Gp** *is a group-2-groupoid.*

Proof. Let $\mathcal{G} = (G_0, G_1, G_2)$ is a 2-category object in **Gp** and μ_0, μ_1, μ_2 be multiplications of groups G_0, G_1, G_2, respectively. Then, we can define a multiplication $\mu\colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ as a 2-functor such that $\mu = \mu_0$ on objects, $\mu = \mu_1$ on 1-morphisms and $\mu = \mu_2$ on 2-morphisms. Similarly, we can define 2-functors $id\colon \mathbf{1} \to \mathcal{G}$ (where $\mathbf{1}$ is the terminal object of **2Cat**, i.e. the one-object discrete 2-category) which picks out an identity object, an identity 1-morphism and an identity 2-morphism and $inv\colon \mathcal{G} \to \mathcal{G}$ picks out inverses for multiplications. Since $\overline{a} = 1_{s(a)}a^{-1}1_{t(a)}$ from [6] and $\overline{\alpha}^v = 1_{s_v(\alpha)}\alpha^{-1}1_{t_v(\alpha)}$, $\overline{\alpha}^h = 1_{1_{s_h(\alpha)}}\alpha^{-1}1_{1_{t_h(\alpha)}}$ from [1], $\mathcal{G}$ is a 2-groupoid. Then, $\mathcal{G}$ is a group object in **2Cat** and so $\mathcal{G}$ is a group-2-groupoid. □

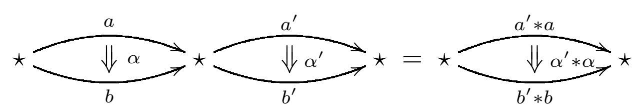
Example 3.2. Every group-groupoid can be thought as a group-2-groupoid in which all 2-morphisms are identities as follows:



It is mentioned that a group-groupoid is a 2-category with a single object [4]. Then, we shall need a different viewpoint on group-groupoids as a special kind of group-2-groupoids:

Proposition 3.3. *A group-2-groupoid with a single object is a group-groupoid in which both groups are necessarily abelian.*

Proof. In this approach, the composition of 1-morphisms and the horizontal composition of 2-morphisms are defined by multiplications of groups as follows:



It is proved in [23] that $a' * a = a' \cdot a = a \cdot a'$. Using similar way, we get

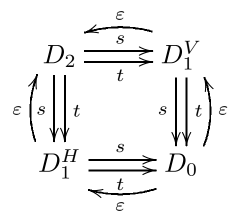
$$\alpha' * \alpha = (\alpha' \cdot 1_e) * (1_e \cdot \alpha) = (\alpha' * 1_e) \cdot (1_e * \alpha) = \alpha' \cdot \alpha$$

and

$$\alpha' \cdot \alpha = (1_e * \alpha') \cdot (\alpha * 1_e) = (1_e \cdot \alpha) * (\alpha' \cdot 1_e) = \alpha \cdot \alpha'.$$

□

A third way to understand group-2-groupoids is to view them as double group-groupoids which are defined in [26] (see also [27]). Recall that a double category is a category object internal to **Cat**. Hence the structure of a double category contains four different but compatible category structures as partially shown in the following diagram



where D_1^H and D_1^V are called horizontal and vertical edge categories, respectively, and D_2 is called the set of squares. For further details, see [12, 14, 15, 20]. The structure of a 2-category may be regarded as a double category in which all vertical morphisms are identities (or D_2 and D_1^H have the same objects) [12, 20]. Therefore, a group-2-groupoid is a special kind of an internal category in the category **GpGd** of group-groupoids.

4. Crossed modules over group-groupoids

In this section, we work on crossed modules over groupoids by replacing such groupoids with group-groupoids. Using the natural equivalence between crossed modules over groupoids and 2-groupoids given in [17], we will prove that there is a categorical equivalence between group-2-groupoids and crossed modules over group-groupoids.

Definition 4.1. Let $\mathcal{G} = (X, G)$ and $\mathcal{H} = (X, H)$ are group-groupoids over the same object set, $\mathcal{H}$ be totally disconnected and $\mathcal{K} = (\mathcal{H}, \mathcal{G}, \partial)$ be a crossed module over $\mathcal{G}$ and $\mathcal{H}$ such that ∂ is a homomorphism of group-groupoids and the following interchange rule holds:

$$(g \bullet h) \cdot (g' \bullet h') = (g \cdot g') \bullet (h \cdot h')$$

where $g, g' \in G, h, h' \in H$. Then $\mathcal{K}$ is called a crossed module over group-groupoids.

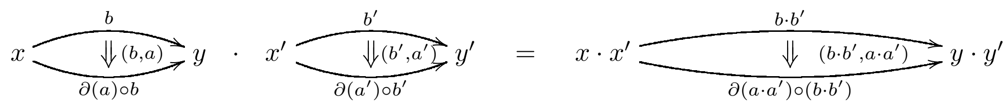
A morphism of crossed modules over group-groupoids is a morphism of crossed modules of groupoids which preserves group structures. Then, we can construct the category of crossed modules over group-groupoids which we denote by **Cmg***.

Theorem 4.2. *The categories* **Cmg*** *and* **Gp2Gd** *are equivalent.*

Proof. The idea of the proof is to show that the functor of 2.2 restricts to an equivalence of categories. Let $\mathcal{A} = (X, A)$ and $\mathcal{B} = (X, B)$ are group-groupoids and $\mathcal{K} = (\mathcal{A}, \mathcal{B}, \partial)$ is a crossed module over $\mathcal{A}$ and $\mathcal{B}$. Then $\theta(\mathcal{K}) = (X, B, B \ltimes A)$ is a group-2-groupoid via the process of the proof 2.2. The group multiplication of 2-morphisms in $\theta(\mathcal{K})$ is defined by

$$(b, a) \cdot (b', a') = (b \cdot b', a \cdot a').$$

We draw such pairs as



Now we will verify that compositions and the group multiplication satisfy the interchange rule.

$$\begin{aligned}
\Big[(b_1,a_1)\circ_h(b,a)\Big]\cdot\Big[(b_1',a_1')\circ_h(b',a')\Big] &= \Big[(b_1\circ b,a_1\circ(b_1\bullet a))\Big]\cdot\Big[(b_1'\circ b',a_1'\circ(b_1'\bullet a'))\Big]\\
&= \Big((b_1\circ b)\cdot(b_1'\circ b'),(a_1\circ(b_1\bullet a)\cdot(a_1'\circ(b_1'\bullet a'))\Big)\\
&= \Big((b_1\cdot b_1')\circ(b\cdot b'),(a_1\cdot a_1')\circ\big((b_1\bullet a)\cdot(b_1'\bullet a')\big)\Big)\\
&= \Big((b_1\cdot b_1')\circ(b\cdot b'),(a_1\cdot a_1')\circ\big((b_1\cdot b_1')\bullet(a\cdot a')\big)\Big)\\
&= (b_1\cdot b_1',a_1\cdot a_1')\circ_h(b\cdot b',a\cdot a')\\
&= \Big[(b_1,a_1)\cdot(b_1',a_1')\Big]\circ_h\Big[(b,a)\cdot(b',a')\Big]
\end{aligned}$$

and

$$\begin{aligned}
\Big[(\partial(a)\circ b,a_2)\circ_v(b,a)\Big]\cdot\Big[(\partial(a')\circ b',a_2')\circ_v(b',a')\Big] &= (b,a_2\circ a)\cdot(b',a_2'\circ a')\\
&= (b\cdot b',(a_2\cdot a_2')\circ(a\cdot a'))\\
&= \Big[\partial(a\cdot a')\circ(b\cdot b'),a_2\cdot a_2'\Big]\circ_v(b\cdot b',a\cdot a')\\
&= \Big[(\partial(a)\circ b,a_2)\cdot(\partial(a')\circ b',a_2')\Big]\circ_v\Big[(b,a)\cdot(b',a')\Big]
\end{aligned}$$

whenever all above compositions are defined.

Now let $\mathcal{G}=(G_0,G_1,G_2)$ be a group-2-groupoid. Then $\gamma(\mathcal{G})$ is a crossed module over groupoids internal to **Gp**. We will verify that the interchange law holds:

$$(f\bullet\alpha)\cdot(f'\bullet\alpha')=(1_f\circ_h\alpha\circ_h 1_{\overline{f}})\cdot(1_{f'}\circ_h\alpha'\circ_h 1_{\overline{f'}})=1_{f\cdot f'}\circ_h(\alpha\cdot\alpha')\circ_h 1_{\overline{f\cdot f'}}=(f\cdot f')\bullet(\alpha\cdot\alpha')$$

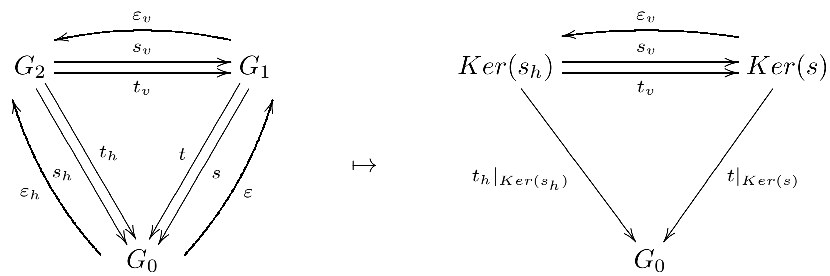
Now we will show that $S_{\mathcal{G}}$ preserves the group multiplication:

$$\begin{aligned}
S_{\mathcal{G}}(\alpha\cdot\alpha') &= (f\cdot f',(\alpha\cdot\alpha')\circ_h 1_{\overline{f\cdot f'}})\\
&= \Big(f\cdot f'\ \ ,\ \ (\alpha\cdot\alpha')\circ_h(1_{\overline{f}}\cdot 1_{\overline{f'}})\Big)\\
&= \Big(f\cdot f'\ \ ,\ \ (\alpha\circ_h 1_{\overline{f}})\cdot(\alpha'\circ_h 1_{\overline{f'}})\Big)\\
&= (f,\alpha\circ_h 1_{\overline{f}})\cdot(f',\alpha'\circ_h 1_{\overline{f'}})\\
&= S_{\mathcal{G}}(\alpha)\cdot S_{\mathcal{G}}(\alpha')
\end{aligned}$$

Other details are straightforward and so are omitted. □

5. Group-2-groupoids as internal categories in Cm

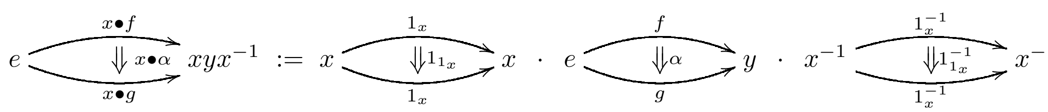
A group-2-groupoid can be also thought as a special case of an internal category in the category **Cm** of crossed modules (see, e.g., [25] and [26] for more details about internal categories in **Cm**). This idea comes from that the structure of a group-2-groupoid contains three compatible group-groupoid structures. Given a group-2-groupoid, we can extract crossed modules as follows:



Then, we obtain an internal groupoid in **Cm**

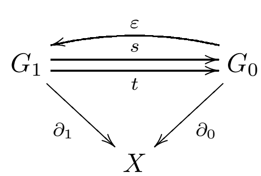
$$(Ker(s_h), G_0) \underset{\mathfrak{t}}{\overset{\mathfrak{s}}{\rightrightarrows}} (Ker(s), G_0), \qquad \epsilon : (Ker(s), G_0) \to (Ker(s_h), G_0)$$

where the structure maps are defined by $\mathfrak{s} = \langle s_v, 1\rangle$, $\mathfrak{t} = \langle t_v, 1\rangle$, $\epsilon = \langle \varepsilon_v, 1\rangle$ as morphisms of crossed modules. Here $\mathfrak{s}, \mathfrak{t}, \epsilon$ are equivariant maps, since $s_v(x \bullet \alpha) = x \bullet s_v(\alpha)$, $t_v(x \bullet \alpha) = x \bullet t_v(\alpha)$ and $\varepsilon_v(x \bullet f) = x \bullet \varepsilon_v(f)$, for all $x \in G_0$ and $\alpha \in Ker(s_h)$. The actions of G_0 on $Ker(s_h)$ and on $Ker(s)$ are drawn in the following diagram:



We denote the category of such internal groupoids in **Cm** by **IGCm**. We know from [25, 26] that internal categories in the category **Cm** of crossed modules are naturally equivalent to crossed squares which in turn should be viewed as a "crossed module of crossed modules". Hence an object of the category **IGCm** can be viewed as a special kind of crossed square.

Let $\mathfrak{G} = (G_0, G_1, X, \partial_0, \partial_1)$ be an object of **IGCm**. Then, the following diagram is commutative.



Let $\mathfrak{G} = (G_0, G_1, X, \partial_0, \partial_1), \mathfrak{G}' = (G'_0, G'_1, X', \partial'_0, \partial'_1)$ be objects of **IGCm**. If (λ_1, λ_2) is an endomorphism of the group-groupoid $G = (G_0, G_1)$, and $\langle\lambda_1, \lambda_0\rangle$, $\langle\lambda_2, \lambda_0\rangle$ are morphisms of crossed modules (G_0, X, ∂_0), (G_1, X, ∂_1), respectively, then $\lambda = (\lambda_2, \lambda_1, \lambda_0)$ is called a morphism of **IGCm**.

Lemma 5.1. *Let* $\mathfrak{G} = (G_0, G_1, X, \partial_0, \partial_1)$ *be an object of* **IGCm**. *Then*

$$x \bullet (\beta \circ \alpha) = (x \bullet \beta) \circ (x \bullet \alpha)$$

for $x \in X, \alpha, \beta \in G_1$ *where* $s(\beta) = t(\alpha)$.

Proof. Let $a \xrightarrow{\alpha} b \xrightarrow{\beta} c$. We know from [6] that $\beta \circ \alpha = \beta \cdot 1_b^{-1} \cdot \alpha$. Then, we get

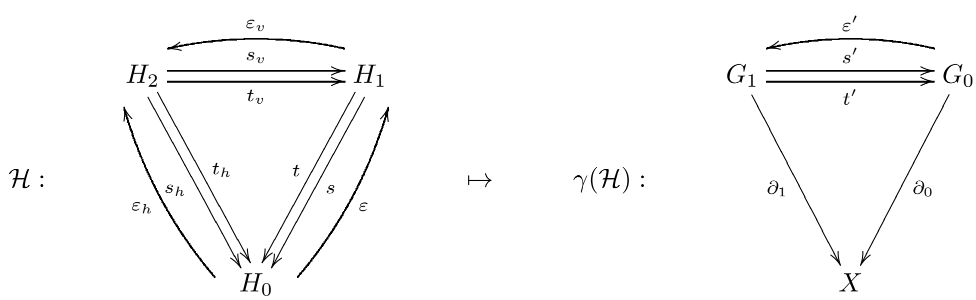
$$\begin{aligned} x \bullet (\beta \circ \alpha) &= x \bullet (\beta \cdot 1_b^{-1} \cdot \alpha) \\ &= (x \bullet \beta) \cdot (x \bullet 1_b^{-1}) \cdot (x \bullet \alpha) \\ &= (x \bullet \beta) \cdot (x \bullet 1_b)^{-1} \cdot (x \bullet \alpha) \\ &= (x \bullet \beta) \cdot 1_{(x \bullet b)}^{-1} \cdot (x \bullet \alpha) \\ &= (x \bullet \beta) \circ (x \bullet \alpha) \end{aligned}$$

□

Example 5.2. Every crossed module $\mathcal{K} = (M, N, \partial)$ over groups is an object of **IGCm** with the discrete groupoid of M where $n \bullet 1_m = 1_{n \bullet m}$ and $\partial_1(1_m) = \partial(m)$.

Theorem 5.3. *There is an equivalence between* **IGCm** *and* **Gp2Gd**.

Proof. A functor $\gamma\colon$ **Gp2Gd** $\to$ **IGCm** is defined in the following way. Let $\mathcal{H} = (H_0, H_1, H_2)$ be a group-2-groupoid. Then $\gamma(\mathcal{H}) = (G_0, G_1, X, \partial_0, \partial_1)$ is an object of **IGCm** where $G_0 = Ker(s)$, $G_1 = Ker(s_h)$, $X = H_0$, $\partial_0 = t\big|_{Ker(s)}$ and $\partial_1 = t_h\big|_{Ker(s_h)}$



with actions $x \bullet f = 1_x \cdot f \cdot 1_x^{-1}$ and $x \bullet \alpha = 1_{1_x} \cdot \alpha \cdot 1_{1_x}^{-1}$, for $x \in X, f \in G_0, \alpha \in G_1$. Now we will verify that s', t', ε' are equivariant maps.

$$s'(x \bullet \alpha) = s'(1_{1_x} \cdot \alpha \cdot 1_{1_x}^{-1}) = s_v(1_{1_x}) \cdot s_v(\alpha) \cdot s_v(1_{1_x}^{-1}) = 1_x \cdot s_v(\alpha) \cdot 1_x^{-1} = x \bullet s'(\alpha),$$

$$t'(x \bullet \alpha) = t'(1_{1_x} \cdot \alpha \cdot 1_{1_x}^{-1}) = t_v(1_{1_x}) \cdot t_v(\alpha) \cdot t_v(1_{1_x}^{-1}) = 1_x \cdot t_v(\alpha) \cdot 1_x^{-1} = x \bullet t'(\alpha)$$

and

$$\varepsilon'(x \bullet f) = \varepsilon'(1_x \cdot f \cdot {1_x}^{-1}) = \varepsilon_v(1_x) \cdot \varepsilon_v(f) \cdot \varepsilon_v({1_x}^{-1}) = 1_{1_x} \cdot \varepsilon_v(f) \cdot 1_{1_x}^{-1} = x \bullet \varepsilon'(f).$$

Let $F = (F_0, F_1, F_2)$ be a morphism of group-2-groupoids. Then $\gamma(F) = (F_2|_{Ker(s_h)}, F_1|_{Ker(s)}, F_0)$ is a morphism of **IGCm**.

Next, we define a functor $\theta\colon \mathbf{IGCm} \to \mathbf{Gp2Gd}$ is an equivalence of categories. Given an object $\mathfrak{G} = (G_0, G_1, X, \partial_0, \partial_1)$ of **IGCm**, we can obtain a group-2-groupoid $\theta(\mathfrak{G}) = \mathcal{H} = (H_0, H_1, H_2)$ where $H_0 = X, H_1 = X \ltimes G_0, H_2 = X \ltimes G_1$ as in the following way. Let $a \xrightarrow{\alpha} b$ be a morphism of $\mathfrak{G}$. Then pairs $x \xrightarrow{(x,a)} \partial_0(a) \cdot x$ and $x \xrightarrow{(x,b)} \partial_0(b) \cdot x$ are obtained as morphisms of the group-groupoid (H_0, H_1), and a pair $x \xrightarrow{(x,\alpha)} \partial_1(\alpha) \cdot x$ is obtained as a morphism of the group-groupoid (H_0, H_2). Since

$$\partial_1(\alpha) \cdot x = \partial_0 s(\alpha) \cdot x = \partial_0(a) \cdot x, \qquad \partial_1(\alpha) \cdot x = \partial_0 t(\alpha) \cdot x = \partial_0(b) \cdot x,$$

then (x, α) can be considered as a 2-morphism as follows:

$$x \underset{(x,b)}{\overset{(x,a)}{\rightrightarrows}} \partial_1(\alpha) \cdot x \qquad \Downarrow (x,\alpha)$$

Let $a \xrightarrow{\alpha} b \xrightarrow{\beta} c$. Then, the vertical composition of (x, α) and (x, β) is defined by

$$(x, \beta) \circ_v (x, \alpha) = (x, \beta \circ \alpha)$$

where the source and the target maps are defined by $s_v(x, \alpha) = (x, s(\alpha))$ and $t_v(x, \alpha) = (x, t(\alpha))$, respectively, and the identity map is defined by $\varepsilon_v(x, a) = (x, 1_a)$. Given morphisms $a \xrightarrow{\alpha} b$ and $a_1 \xrightarrow{\alpha_1} b_1$, we obtain pairs (x, α), $(\partial_1(\alpha) \cdot x, \alpha_1)$ and we define their horizontal composite by

$$(\partial_1(\alpha) \cdot x, \alpha_1) \circ_h (x, \alpha) = (x, \alpha_1 \cdot \alpha)$$

where the source and the target maps are defined by $s_h(x, \alpha) = x, t_h(x, \alpha) = \partial_1(\alpha) \cdot x$, respectively, and the identity map is defined by $\varepsilon_h(x) = (x, 1_e)$. Clearly the vertical composition and the horizontal composition satisfy the usual interchange rule. The product of (x, α) and (x', α') is written by

$$(x, \alpha) \cdot (x', \alpha') = (x \cdot x', \alpha \cdot (x \bullet \alpha'))$$

for $a \xrightarrow{\alpha} b$ and $a' \xrightarrow{\alpha'} b'$.

If $\lambda = (\lambda_2, \lambda_1, \lambda_0)$ is a morphism of $\mathfrak{G}$, then $\theta(\lambda) = (\lambda_0, \lambda_0 \times \lambda_1, \lambda_0 \times \lambda_2)$ is morphism of $\theta(\mathfrak{G})$.

A natural equivalence $S\colon 1_{\mathbf{Gp2Gd}} \to \theta\gamma$ is defined with a map $S_{\mathcal{G}}\colon \mathcal{G} \to \theta\gamma(\mathcal{G})$ which is defined such that to be the identity on objects, $S_{\mathcal{G}}(f) =$

$(x, f\cdot 1_x^{-1})$ and $S_{\mathcal{G}}(\alpha) = (x, \alpha\cdot 1_{1_x}^{-1})$ for $f \in G_1, \alpha \in G_2$ where $x = s(f) = s_h(\alpha)$. Clearly $S_{\mathcal{G}}$ is an isomorphism and preserves the group operations and compositions as follows:

$$\begin{aligned}
S_{\mathcal{G}}(\alpha)\cdot S_{\mathcal{G}}(\alpha') &= (x, \alpha\cdot 1_{1_x}^{-1})\cdot(x', \alpha'\cdot 1_{1_{x'}}^{-1})\\
&= \Big(x\cdot x', \alpha\cdot 1_{1_x}^{-1}\cdot(x\bullet(\alpha'\cdot 1_{1_{x'}}^{-1}))\Big)\\
&= \Big(x\cdot x', \alpha\cdot 1_{1_x}^{-1}\cdot 1_{1_x}\cdot\alpha'\cdot 1_{1_{x'}}^{-1}\cdot 1_{1_x}^{-1}\Big)\\
&= \big(x\cdot x', \alpha\cdot\alpha'\cdot 1_{1_{xx'}}^{-1}\big)\\
&= S_{\mathcal{G}}(\alpha\cdot\alpha')
\end{aligned}$$

where $s(\alpha) = x,\ s(\alpha') = x'$,

$$S_{\mathcal{G}}(\delta\circ_h\alpha) = S_{\mathcal{G}}(\delta\cdot 1_{1_y}^{-1}\cdot\alpha) = (x, \delta\cdot 1_{1_y}^{-1}\cdot\alpha\cdot 1_{1_x}^{-1}) = (y, \delta\cdot 1_{1_y}^{-1})\circ_h(x, \alpha\cdot 1_{1_x}^{-1}) = S_{\mathcal{G}}(\delta)\circ_h S_{\mathcal{G}}(\alpha)$$

where $t(\alpha) = s(\delta) = y$ and

$$\begin{aligned}
S_{\mathcal{G}}(\beta)\circ_v S_{\mathcal{G}}(\alpha) &= (x, \beta\cdot 1_{1_x}^{-1})\circ_v(x, \alpha\cdot 1_{1_x}^{-1})\\
&= \Big(x, (\beta\cdot 1_{1_x}^{-1})\circ_v(\alpha\cdot 1_{1_x}^{-1})\Big)\\
&= \Big(x, (\beta\circ_v\alpha)\cdot(1_{1_x}^{-1}\circ_v 1_{1_x}^{-1})\Big)\\
&= \Big(x, (\beta\circ_v\alpha)\cdot 1_{1_x}^{-1}\Big)\\
&= S_{\mathcal{G}}(\beta\circ_v\alpha)
\end{aligned}$$

where $s_v(\beta) = t_v(\alpha)$.

To define a natural equivalence $T\colon 1_{\mathbf{IGCm}} \to \gamma\theta$, a map $T_{\mathfrak{G}}$ is defined such that to be identity on X, $T_{\mathfrak{G}}(a) = (e, a)$ for $a \in G_0$ and $T_{\mathfrak{G}}(\alpha) = (e, \alpha)$ for $\alpha \in G_1$. Obviously $T_{\mathfrak{G}}$ is an isomorphism and preserves the composition and the group multiplication as follows:

$$T_{\mathfrak{G}}(\beta\circ\alpha) = (e, \beta\circ\alpha) = (e, \beta)\circ(e, \alpha) = T_{\mathfrak{G}}(\beta)\circ T_{\mathfrak{G}}(\alpha)$$

$$T_{\mathfrak{G}}(\alpha)\cdot T_{\mathfrak{G}}(\alpha') = (e, \alpha)\cdot(e, \alpha') = (e, \alpha\cdot(e\bullet\alpha')) = (e, \alpha\cdot\alpha') = T_{\mathfrak{G}}(\alpha\cdot\alpha').$$

Other details are straightforward and so are omitted. □

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