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$[\ell_p]_{e.r}$ Euler-Riesz Difference Sequence Spaces

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Abstract. Başar and Braha [1], introduced the sequence spaces $\breve{\ell}_\infty$, $\breve{c}$ and $\breve{c}_0$ of Euler-Cesáro bounded, convergent and null difference sequences and studied their some properties. Then, in [2], we introduced the sequence spaces $[\ell_\infty]_{e.r}$, $[c]_{e.r}$ and $[c_0]_{e.r}$ of Euler-Riesz bounded, convergent and null difference sequences by using the composition of the Euler mean E_1 and Riesz mean R_q with backward difference operator Δ. The main purpose of this study is to introduce the sequence space $[\ell_p]_{e.r}$ of Euler-Riesz $p-$absolutely convergent series, where $1 \le p < \infty$, difference sequences by using the composition of the Euler mean E_1 and Riesz mean R_q with backward difference operator Δ. Furthermore, the inclusion $\ell_p \subset [\ell_p]_{e.r}$ hold, the basis of the sequence space $[\ell_p]_{e.r}$ is constucted and $\alpha-$, $\beta-$ and $\gamma-$duals of the space are determined. Finally, the classes of matrix transformations from the $[\ell_p]_{e.r}$ Euler-Riesz difference sequence space to the spaces ℓ_∞, c and c_0 are characterized. We devote the final section of the paper to examine some geometric properties of the space $[\ell_p]_{e.r}$.

Key Words: Composition of summability methods, Riesz mean of order one, Euler mean of order one, backward difference operator, sequence space, BK space, Schauder basis, $\beta-$duals, matrix transformations.

AMS Subject Classifications: 40C05, 40A05, 46A45

1 Preliminaries, background and notation

By a sequence space, we understand a linear subspace of the space $w = \mathbb{C}^{\mathbb{N}}$ of all complex sequnces which contains ϕ, the set of all finitely non-zero sequences, where $\mathbb{N} = \{0, 1, \cdots\}$. We shall write ℓ_∞, c and c_0 for the spaces of all bounded, convergent and null sequences, respectively. Also by bs, cs, ℓ_1 and ℓ_p, we denote the spaces of all bounded, convergent, absolutely and $p-$absolutely convergent series, respectively, where $1 < p < \infty$.

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We shall assume throughout unless stated otherwise that $p,q>1$ with $p^{-1}+q^{-1}=1$ and $0<r<1$, and use the convention that any term with negative subscript is equal to naught.

Let λ,μ be two sequence spaces and $A=(a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n,k\in\mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by writing $A:\lambda\to\mu$, if for every sequence $x=(x_k)\in\lambda$ the sequence $Ax=\{(Ax)_n\}$, the $A-$transform of x, is in μ; where

$$(Ax)_n=\sum_k a_{nk}x_k, \quad (n\in\mathbb{N}). \tag{1.1}$$

By (λ,μ), we denote the class of all matrices A such that $A:\lambda\to\mu$. Thus, $A\in(\lambda,\mu)$ if and only if the series on the right hand side of (1.1) converges for each $n\in\mathbb{N}$ and every $x\in\lambda$, and we have $Ax=\{(Ax)_n\}_{n\in\mathbb{N}}\in\mu$ for all $x\in\lambda$. A sequence x is said to be $A-$summable to α if Ax converges to α which is called the $A-$limit of x.

Let X be a sequence space and A be an infinite matrix. The sequence space

$$X_A=\{x=(x_k)\in w: Ax\in X\} \tag{1.2}$$

is called the domain of A in X which is a sequence space.

A sequence space λ with a linear topology is called a $K-$ space provided each of the maps $p_i:\lambda\to\mathbb{C}$ defined by $p_i(x)=x_i$ is continuous for all $i\in\mathbb{N}$. A $K-$ space is called an $FK-$ space provided λ is a complete linear metric space. An $FK-$ space whose topology is normal is called a $BK-$ space. If a normed sequence space λ contains a sequence (b_n) with the property that for every $x\in\lambda$ there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty}||x-(\alpha_0b_0+\alpha_1b_1+\cdots+\alpha_nb_n)||=0,$$

then (b_n) is called a Schauder basis (or briefly basis) for λ. The series $\sum\alpha_kb_k$ which has the sum x is then called the expansion of x with respect to (b_n), and written as $x=\sum\alpha_kb_k$.

A matrix $A=(a_{nk})$ is called a triangle if $a_{nk}=0$ for $k>n$ and $a_{nn}\neq 0$ for all $n\in\mathbb{N}$. It is trivial that $A(Bx)=(AB)x$ holds for the triangle matrices A,B and a sequence x. Further, a triangle matrix U uniquely has an inverse $U^{-1}=V$, which is also a triangle matrix. Then, $x=U(Vx)=V(Ux)$ holds for all $x\in w$.

Let us give the definition of some triangle limitation matrices which are needed in the text. Δ denotes the backward difference matrix $\Delta=(\Delta_{nk})$ and $\Delta'=(\Delta'_{nk})$ denotes the transpose of the matrix Δ, the forward difference matrix, which are defined by

$$\Delta_{nk}=\begin{cases}(-1)^{n-k}, & n-1\le k\le n,\\ 0, & 0\le k<n-1 \quad \text{or} \quad k>n,\end{cases}$$
$$\Delta'_{nk}=\begin{cases}(-1)^{n-k}, & n\le k\le n+1,\\ 0, & 0\le k<n \quad \text{or} \quad k>n+1,\end{cases}$$

for all $k,n\in\mathbb{N}$; respectively.

Then, let us define the Euler mean $E_1 = (e_{nk})$ of order one and Riesz mean $R_q = (r_{nk})$

$$e_{nk} = \begin{cases} \dfrac{\binom{n}{k}}{2^n}, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases} \qquad r_{nk} = \begin{cases} \dfrac{q_k}{Q_n}, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$ and where (q_k) be a sequence of positive numbers and $Q_n = \sum_{k=0}^{n} q_k$. Their inverses $E_1^{-1} = (g_{nk})$ and $R_q^{-1} = (h_{nk})$ are given by

$$g_{nk} = \begin{cases} \binom{n}{k}(-1)^{n-k}2^k, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases} \qquad h_{nk} = \begin{cases} (-1)^{n-k}\dfrac{Q_k}{q_k}, & n-1 \leq k \leq n, \\ 0, & \text{otherwise}, \end{cases}$$

for all $k, n \in \mathbb{N}$.

We define the matrix $\tilde{B} = (\tilde{b}_{nk})$ by the composition of the matrices E_1, R_q and Δ as

$$\tilde{b}_{nk} = \begin{cases} \dfrac{\binom{n}{k} q_k}{2^n Q_n}, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases} \tag{1.3}$$

for all $k, n \in \mathbb{N}$.

In the literature, the notion of difference sequence spaces was introduced by Kızmaz [8] as

$$X(\Delta) = \{x = (x_k) \in w : \Delta' x = (x_k - x_{k+1}) \in X\}$$

for $X \in \{\ell_\infty, c, c_0\}$. The difference space bv_p, consisting of all sequences $x = (x_k)$ such that $\Delta x = (x_k - x_{k-1})$ is in the sequence space ℓ_p, was studied in the case $0 < p < 1$ by Altay and Başar [22] and in the case $1 \leq p \leq \infty$ by Başar and Altay [9], and Çolak et al. [4]. Kirişçi and Başar [10] have introduced and studied the generalized difference sequence space

$$\hat{X} = \{x = (x_k) \in w : B(r,s)x \in X\},$$

where X denotes any of the spaces ℓ_∞, c, c_0 and ℓ_p with $1 \leq p < \infty$, and $B(r,s)x = (s.x_{k-1} + r.x_k)$ with $r, s \in \mathbb{R} \setminus \{0\}$. Following Kirişçi and Başar [10], Sönmez [11] have been examined the sequence space $X(B)$ as the set of all sequences whose $B(r,s,t)-$ trasforms are in the space $X \in \{\ell_\infty, c, c_0, \ell_p\}$, where $B(r,s,t)$ denotes the triple band matrix $B(r,s,t) = \{b_{nk}\{r,s,t\}\}$ defined by

$$b_{nk}\{r,s,t\} = \begin{cases} r, & n = k, \\ s, & n = k+1, \\ t, & n = k+2, \\ 0, & \text{otherwise}, \end{cases}$$

for all $k, n \in \mathbb{N}$ and $r, s, t \in \mathbb{R} \setminus \{0\}$. Quite recently, Başar has studied the spaces $\tilde{\ell}_p$ of $p-$absolutely $\tilde{B}-$summable sequences, in [6]. The reader can also review these references to get more detailed information [12–14].

Then, as a natural continuation of Başar [6], Başar and Braha [1] introduce the spaces $\breve{\ell}_\infty$, $\breve{c}$ and $\breve{c}_0$ of Euler-Cesáro bounded, convergent and null difference sequences by using the composition of the Euler mean E_1 and Cesáro mean C_1 of order one with backward difference operator Δ. In [2], we introduced the $[\ell_\infty]_{e.r}$, $[c]_{e.r}$ and $[c_0]_{e.r}$ of Euler-Riesz bounded, convergent and null difference sequences by using the composition of the Euler mean E_1 and Riesz mean R_q with backward difference operator Δ and prove that the inclusions $\ell_\infty \subset [\ell_\infty]_{e.r}$, $c \subset [c]_{e.r}$ and $c_0 \subset [c_0]_{e.r}$ strictly hold. Furthermore, we investigated some properties and compute $\alpha-$, $\beta-$ and $\gamma-$ duals of these spaces. Afterwards, we characterized of some matrix classes of Euler-Riesz sequence spaces.

In the present paper, we introduce the $[\ell_p]_{e.r}$ of Euler-Riesz bounded, convergent and null difference sequence by using the composition of the Euler mean E_1 and Riesz mean R_1 of order one with backward difference operator Δ. Furthermore, we investigate some properties and compute $\alpha-$, $\beta-$ and $\gamma-$ duals of these space. Afterwards, we characterize of some matrix classes of Euler-Riesz sequence space. We devote the final section of the paper to examine some geometric properties of the space $[\ell_p]_{e.r}$

2 The Euler-Riesz sequence space

In this section, we shall give a new sequence space and we shall investigate its some properties:

$$[\ell_p]_{e.r} = \left\{ x = (x_k) \in w : \sum_{n=0}^{\infty} \left| \sum_{k=0}^{n} \frac{\binom{n}{k} q_k}{2^n Q_n} x_k \right|^p < \infty \right\}.$$

With the notation (1.2), we may redefine the space $[\ell_p]_{e.r}$ as fallows:

$$[\ell_p]_{e.r} = (\ell_p)_{\tilde{B}}. \tag{2.1}$$

Define the sequence $y = (y_k)$, which will be frequently used, as the $\tilde{B}-$transform of a sequence $x = (x_k)$, i.e.,

$$y_k = \sum_{j=0}^{k} \frac{\binom{k}{j} q_j}{2^k Q_k} x_j, \quad k \in \mathbb{N}. \tag{2.2}$$

Throughout the text, we suppose that the sequences $x = (x_k)$ and $y = (y_k)$ are connected with the relation (2.2). One can obtain by a straightforward calculation from (2.2) that

$$x_k = \frac{1}{q_k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} 2^j Q_j y_j, \quad k \in \mathbb{N}. \tag{2.3}$$

Theorem 2.1. *The set $[\ell_p]_{e.r}$ is linear space with coordinatewise addition and scalar multiplication, and it is a BK$-$space with norm $||x||_{[\ell_p]_{e.r}} = ||\tilde{B}x||_p$.*

Proof. The proof of the first part of the theorem is a routine verification, and so we omit it. Furthermore, since (2.2) holds, ℓ_p is $BK-$space with respect to its natural norm, and the matrix $\tilde{B}$ is a triangle, Theorem 4.3.2 of Wilansky [16] implies that the spaces $[\ell_p]_{e.r}$ is $BK-$space. □

Therefore, one can easily check that the absolute property does not hold on the space $[\ell_p]_{e.r}$, because $||x||_{[\ell_p]_{e.r}} \neq |||x|||_{[\ell_p]_{e.r}}$ for at least one sequence in the space $[\ell_p]_{e.r}$, where $|x| = (|x_k|)$. This says that $[\ell_p]_{e.r}$ is the sequence space of nonabsolute type.

Theorem 2.2. *$[\ell_p]_{e.r}$ is linearly isomorphic to the space ℓ_p, i.e., $[\ell_p]_{e.r} \cong \ell_p$.*

Proof. To prove this theorem, we should show the existence of a linear bijection between the spaces $[\ell_p]_{e.r}$ and ℓ_p. Consider the transformation S from $[\ell_p]_{e.r}$ to ℓ_p by $y = Sx = \tilde{B}x$. The linearity of S is clear. Further, it is obvious that $x = \theta$ whenever $Sx = \theta$ and hence S is injective, where $\theta = (0, 0, 0, \cdots)$.

Let us take any $y \in \ell_p$ and define the sequence $x = \{x_n\}$ by

$$x_n = \frac{1}{q_n}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}2^k Q_k y_k \quad \text{for all } n \in \mathbb{N}.$$

Then, we obtain in the case of $1 \leq p < \infty$ that

$$\begin{aligned}
||x||_{[\ell_p]_{e.r}} &= \left[\sum_{n=0}^{\infty}\left|\sum_{k=0}^{n}\frac{\binom{n}{k}q_k}{2^n Q_n}x_k\right|^p\right]^{1/p} \\
&= \left[\sum_{n=0}^{\infty}\left|\sum_{k=0}^{n}\frac{\binom{n}{k}q_k}{2^n Q_n}\frac{1}{q_k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}2^j Q_j y_j\right|^p\right]^{1/p} \\
&= \left(\sum_{n}\left|\sum_{k=n}^{\infty}\delta_{nk}y_k\right|^p\right)^{1/p} = ||y||_{\ell_p} < \infty.
\end{aligned}$$

Consequently, we see from here that S is surjective. Hence, S is a linear bijection which therefore says us that the spaces $[\ell_p]_{e.r}$ and ℓ_p are linearly isomorphic, as desired. □

Theorem 2.3. *The inclusion $\ell_p \subset [\ell_p]_{e.r}$ strictly holds for $1 \leq p < \infty$.*

Proof. To prove the validity of the inclusion $\ell_p \subset [\ell_p]_{e.r}$ for $1 \leq p < \infty$, it suffices to show the existence of a number $K > 0$ such that $||x||_{[\ell_p]_{e.r}} \leq K \cdot ||x||_{\ell_p}$ for every $x \in \ell_p$.

Let us take any $x \in \ell_p$. Then we obtain, with the notation of (2.2), by applying the Hölder's inequality for $1 < p < \infty$ that

$$\begin{aligned}
|y_k|^p &= \left|\sum_{j=0}^{k}\frac{\binom{k}{j}q_j}{2^k Q_k}x_j\right|^p \leq \left|\sum_{j=0}^{k}\frac{\binom{k}{j}Q_k}{2^k Q_k}x_j\right|^p = \left|\sum_{j=0}^{k}\frac{\binom{k}{j}}{2^k}x_j\right|^p \\
&\leq \left[\sum_{j=0}^{k}\frac{\binom{k}{j}}{2^k}|x_j|^p\right] \times \left[\sum_{j=0}^{k}\frac{\binom{k}{j}}{2^k}\right]^{p-1} = \sum_{j=0}^{k}\frac{\binom{k}{j}}{2^k}|x_j|^p. \qquad (2.4)
\end{aligned}$$

Using (2.4), we have that

$$\sum_k |y_k|^p \leq \sum_k \sum_{j=0}^{k} \frac{\binom{k}{j}}{2^k} |x_j|^p \leq \sum_k |x_k|^p \sum_{j=0}^{k} \frac{\binom{k}{j}}{2^k} = \sum_k |x_k|^p,$$

which yields us that

$$||x||_{[\ell_p]_{e.r}} \leq ||x||_{\ell_p} \tag{2.5}$$

for $1 < p < \infty$, as expected. Besides, let us consider the sequence $u = \{u_k^{(n)}\}$ defined by

$$u_k^{(n)} = \Big\{0,0,0,\cdots, \underbrace{\frac{Q_n}{q_n}}_{n-\text{th}}, \cdots\Big\}$$

for all $n \in \mathbb{N}$. Then, we have

$$(\tilde{B}u)_n = \sum_{k=0}^{n} \frac{\binom{n}{k} q_k}{2^n Q_n} u_k^{(n)} = \frac{1}{2^n}.$$

For every $n = 0, 1, \cdots$, $(\tilde{B}u)_n \in \ell_p$, but the sequence $u = \{u_k^{(n)}\}$ is not in ℓ_p. By the similar discussions, it may be easily proved that the inequality (2.5) also holds in the case $p = 1$ and so we omit the detail. This completes the proof. □

Since the isomorphism S, defined in Theorem 2.1, is surjective, the inverse image of the basis of the spaces ℓ_p is the basis of the new space $[\ell_p]_{e.r}$. Therefore, we have the following theorem without proof.

Theorem 2.4. *Define a sequence $b^{(k)} = \{b_n^{(k)}\}_{n\in\mathbb{N}}$ of elements of the space $[\ell_p]_{e.r}$ for every fixed $k \in \mathbb{N}$ by*

$$b_n^{(k)} = \begin{cases} \dfrac{\binom{n}{k}(-1)^{n-k} 2^k Q_k}{q_n}, & 0 \leq k < n, \\ 0, & k \geq n. \end{cases}$$

Let $\lambda_k = (\tilde{B}x)_k$ for all $k \in \mathbb{N}$. Then, the sequence $\{b^{(k)}\}_{k\in\mathbb{N}}$ is a basis for the space $[\ell_p]_{e.r}$ and any $x \in [\ell_p]_{e.r}$ has a unique representation of the form

$$x = \sum_k \lambda_k b^{(k)}.$$

Remark 2.1. It is well known that every Banach space X with a Schauder basis is separable.

From Theorem 2.4 and Remark 2.1, we can give following corollary:

Corollary 2.1. *The spaces $[\ell_p]_{e.r}$ is separable.*

3 Duals of the new sequence spaces

In this section, we state and prove the theorems determining the $\alpha-$, $\beta-$ and $\gamma-$ duals of the sequence space $[\ell_p]_{e.r}$.

The set $S(\lambda,\mu)$ defined by

$$S(\lambda,\mu)=\{z=(z_k)\in w: xz=(x_kz_k)\in\mu \text{ for all } x=(x_k)\in\lambda\} \tag{3.1}$$

is called the multiplier space of the sequence spaces λ and μ. One can easily observe for a sequence space ν with $\lambda\supset\nu\supset\mu$ that the inclusions

$$S(\lambda,\mu)\subset S(\nu,\mu) \quad\text{and}\quad S(\lambda,\mu)\subset S(\lambda,\nu)$$

hold. With the notation of (3.1), the alpha-, beta- and gamma-duals of a sequence space λ, which are respectively denoted by λ^α, λ^β and λ^γ are defined by

$$\lambda^\alpha=S(\lambda,\ell_1),\quad \lambda^\beta=S(\lambda,cs)\quad\text{and}\quad \lambda^\gamma=S(\lambda,bs).$$

For to give the $\alpha-$, $\beta-$ and $\gamma-$ duals of the space $[\ell_p]_{e.r}$ of non-absolute type, we need the following Lemma;

Lemma 3.1 ([18]). *$A\in(\ell_p:\ell_1)$ if and only if*

$$\sup_{K\in\mathcal{F}}\sum_k\left|\sum_{n\in K}a_{nk}\right|^q<\infty,\quad (1<p\leq\infty).$$

Here and in what follows, we denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$.

Lemma 3.2 ([18]). *$A\in(\ell_p:c)$ if and only if*

$$\lim_{n\to\infty}a_{nk} \text{ exists for each } k\in\mathbb{N}, \tag{3.2a}$$

$$\sup_{n\in\mathbb{N}}\sum_k|a_{nk}|^q<\infty,\quad (1<p<\infty). \tag{3.2b}$$

Lemma 3.3 ([18]). *$A\in(\ell_p:\ell_\infty)$ if and only if* (3.2b) *holds.*

Now, we may give the theorems determining the $\alpha-$, $\beta-$ and $\gamma-$duals of the Euler-Riesz sequence space $[\ell_p]_{e.r}$.

Theorem 3.1. *Define the set a_q as follows:*

$$a_q=\left\{a=(a_k)\in w:\sup_{K\in\mathcal{F}}\sum_{k=0}^{\infty}\left|\sum_{n\in K}\binom{n}{k}(-1)^{n-k}2^k\frac{a_n}{q_n}Q_k\right|^q<\infty\right\}.$$

Then, $\{[\ell_p]_{e.r}\}^\alpha=a_q$.

Proof. We chose the sequence $a = (a_k) \in w$. We can easily derive that with the (2.3) that

$$a_n x_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} 2^k \frac{a_n}{q_n} Q_k y_k = (By)_n, \quad (n \in \mathbb{N}), \tag{3.3}$$

where $B = (b_{nk})$ is defined by the formula

$$b_{nk} = \begin{cases} \binom{n}{k}(-1)^{n-k} 2^k \frac{a_n}{q_n} Q_k, & (0 \le k \le n), \\ 0, & (k > n), \end{cases} \quad (n,k \in \mathbb{N}). \tag{3.4}$$

It follows from (3.3) that $ax = (a_n x_n) \in \ell_1$ whenever $x \in [\ell_p]_{e.r}$ if and only if $By \in \ell_1$ whenever $y \in c_0$. This gives the result that $\{[\ell_p]_{e.r}\}^{\alpha} = a_q$. ☐

Theorem 3.2. *The matrix $D(r) = (d_{nk})$ is defined by*

$$d_{nk} = \begin{cases} \sum_{j=k}^{n} \binom{j}{k} (-1)^{j-k} 2^k \frac{a_j}{q_j} Q_k, & (0 \le k \le n), \\ 0, & (k > n), \end{cases} \tag{3.5}$$

for all $k, n \in \mathbb{N}$. Then, $\{[\ell_p]_{e.r}\}^{\beta} = b_1 \cap b_2$ where

$$b_1 = \left\{ a = (a_k) \in w : \lim_{n\to\infty} d_{nk} = \alpha_k \right\},$$

$$b_2 = \left\{ a = (a_k) \in w : \sup_{n\in\mathbb{N}} \sum_k |d_{nk}|^q < \infty \right\}.$$

Proof. We give the proof for the space $[\ell_p]_{e.r}$. Consider the equation

$$\begin{aligned} \sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} \left[\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} 2^j \frac{1}{q_k} Q_j y_j \right] a_k \\ &= \sum_{k=0}^{n} \left[\sum_{j=k}^{n} \binom{k}{j} (-1)^{k-j} 2^j \frac{a_k}{q_k} Q_j \right] y_k = (Dy)_n, \end{aligned} \tag{3.6}$$

where $D = (d_{nk})$ defined by (3.4).

Thus, we deduce by with (3.6) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in [\ell_p]_{e.r}$ if and only if $Dy \in c$ whenever $y = (y_k) \in \ell_p$. Therefore, we derive from (3.2a) and (3.2b) that

$$\lim_{n\to\infty} d_{nk} \text{ exists for each } k \in \mathbb{N},$$

$$\sup_{n\in\mathbb{N}} \sum_{k=0}^{n} |d_{nk}|^q < \infty, \quad (1 < p < \infty),$$

which shows that $\{[\ell_p]_{e.r}\}^{\beta} = b_1 \cap b_2$. ☐

Theorem 3.3. $\{[\ell_p]_{e.r}\}^{\gamma} = b_2$.

Proof. This is obtained in the similar way used in the proof of Theorem 3.2. ☐

4 Matrix transformations related to the new sequence spaces

In this section, we characterize the matrix transformations from the space $[\ell_p]_{e.r}$ into any given sequence space μ and from the sequence space μ into the space $[\ell_p]_{e.r}$.

We known that, if $[\ell_p]_{e.r} \cong \ell_p$, we can say: The equivalence $x \in [\ell_p]_{e.r}$, if and only if $y \in \ell_p$ holds.

In what follows, for brevity, we write,

$$\tilde{a}_{nk} =: \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} 2^k \frac{Q_k}{q_n} a_{nk}$$

for all $k, n \in \mathbb{N}$.

Theorem 4.1. *Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $E = (e_{nk})$ are connected with the relation*

$$e_{nk} =: \tilde{a}_{nk} \tag{4.1}$$

for all $k, n \in \mathbb{N}$ and μ be any given sequence space. Then, $A \in ([\ell_p]_{e.r} : \mu)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{[\ell_p]_{e.r}\}^\beta$ for all $n \in \mathbb{N}$ and $E \in (\ell_p : \mu)$.

Proof. Let μ be any given sequence space. Suppose that (4.1) holds between $A = (a_{nk})$ and $E = (e_{nk})$, and take into account that the space $[\ell_p]_{e.r}$ and ℓ_p are linearly isomorphic.

Let $A \in ([\ell_p]_{e.r} : \mu)$ and take any $y = (y_k) \in \ell_p$. Then, $E\tilde{B}$ exists and $\{a_{nk}\}_{k\in\mathbb{N}} \in b_1 \cap b_2$ which yields that $\{e_{nk}\}_{k\in\mathbb{N}} \in \ell_1$ for each $n \in \mathbb{N}$. Hence, Ey exists and thus

$$\sum_k e_{nk} y_k = \sum_k a_{nk} x_k$$

for all $n \in \mathbb{N}$.

We have that $Ey = Ax$ which leads us to the consequence $E \in (\ell_p : \mu)$.

Conversely, let $\{a_{nk}\}_{k\in\mathbb{N}} \in \{[\ell_p]_{e.r}\}^\beta$ for each $n \in \mathbb{N}$ and $E \in (\ell_p : \mu)$ hold, and take any $x = (x_k) \in [\ell_p]_{e.r}$. Then, Ax exists. Therefore, we obtain from the equality

$$\sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} \left[\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} 2^j \frac{Q_j}{q_k} a_{kj} \right] y_k$$

for all $n \in \mathbb{N}$, that $Ey = Ax$ and this shows that $A \in ([\ell_p]_{e.r} : \mu)$. This completes the proof. □

Theorem 4.2. *Suppose that the elements of the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ are connected with the relation*

$$b_{nk} =: \sum_{j=0}^{k} \frac{\binom{k}{j} q_j}{2^k Q_k} a_{jk} \quad \text{for all } k, n \in \mathbb{N}. \tag{4.2}$$

Let μ be any given sequence space. Then, $A \in (\mu : [\ell_p]_{e.r})$ if and only if $B \in (\mu : \ell_p)$.

Proof. Let $z=(z_k)\in\mu$ and consider the following equality.

$$\sum_{k=0}^{m} b_{nk}z_k=\sum_{j=0}^{k}\frac{\binom{k}{j}q_j}{2^kQ_k}\left(\sum_{k=0}^{m}a_{jk}z_k\right)\quad\text{for all } m,n\in\mathbb{N},$$

which yields as $m\to\infty$ that $(Bz)_n=\{\tilde{B}(Az)\}_n$ for all $n\in\mathbb{N}$. Therefore, one can observe from here that $Az\in[\ell_p]_{e.r}$ whenever $z\in\mu$ if and only if $Bz\in\ell_p$ whenever $z\in\mu$. This completes the proof. □

The following results were taken from Stieglitz and Tietz [18]:

$$\lim_{n\to\infty}a_{nk}=0, \tag{4.3a}$$

$$\sup_K\sum_n\left|\sum_{k\in K}a_{nk}\right|^p<\infty. \tag{4.3b}$$

Lemma 4.1. *Let $A=(a_{nk})$ be an infinite matrix. Then*

(i) $A=(a_{nk})\in(c_0:\ell_p)=(c:\ell_p)=(\ell_\infty:\ell_p)$ *if and only if* (4.3b) *holds.*

(ii) $A=(a_{nk})\in(\ell_p:c_0)$ *if and only if* (3.2b) *and* (4.3a) *hold.*

(iii) $A=(a_{nk})\in(\ell_p:c)$ *if and only if* (3.2a) *and* (3.2b) *hold.*

(iv) $A=(a_{nk})\in(\ell_p:\ell_\infty)$ *if and only if* (3.2b) *holds.*

Now, we can give the following results:

Corollary 4.1. *Let $A=(a_{nk})$ be an infinite matrix. The following statements hold:* $(1<p<\infty)$

(i) $A\in([\ell_p]_{e.r}:\ell_\infty)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}}\in\{[\ell_p]_{e.r}\}^\beta$ *for all* $n\in\mathbb{N}$ *and* (3.2a) *holds with* $\tilde{a}_{nk}$ *instead of* a_{nk}.

(ii) $A\in([\ell_p]_{e.r}:c)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}}\in\{[\ell_p]_{e.r}\}^\beta$ *for all* $n\in\mathbb{N}$ *and* (3.2a) *and* (3.2b) *hold with* $\tilde{a}_{nk}$ *instead of* a_{nk}.

(iii) $A\in([\ell_p]_{e.r}:c_0)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}}\in\{[\ell_p]_{e.r}\}^\beta$ *for all* $n\in\mathbb{N}$ *and* (3.2a) *and* (4.3a) *hold with* $\tilde{a}_{nk}$ *instead of* a_{nk}.

Corollary 4.2. *Let $A=(a_{nk})$ be an infinite matrix. The following statements hold:* $A=(a_{nk})\in(c_0:[\ell_p]_{e.r})=(c:[\ell_p]_{e.r})=(\ell_\infty:[\ell_p]_{e.r})$ *if and only if* (4.3b) *holds with* b_{nk} *instead of* a_{nk}.

5 Some geometric properties of the space $[\ell_p]_{e.r}$

In this section, we study some geometric properties of the space $[\ell_p]_{e.r}$.

A Banach space X is said to have the Banach-Saks property if every bounded sequence (x_n) in X admits a subsequence (z_n) such that the sequence $\{t_k(z)\}$ is convergent in the norm in X [24], where

$$t_k(z) = \frac{1}{k+1}(z_0 + z_1 + \cdots + z_k), \quad (k \in \mathbb{N}). \tag{5.1}$$

A Banach space X is said to have the weak Banach-Saks property whenever given any weakly null sequence $(x_n) \subset X$ and there exists a subsequence (z_n) of (x_n) such that the sequence $\{t_k(z)\}$ strongly convergent to zero.

In [25], García-Falset introduce the following coefficient:

$$R(X) = \sup\left\{\liminf_{n\to\infty} ||x_n - x|| : (x_n) \subset B(X),\ x_n \underrightarrow{w} 0,\ x \in B(X)\right\}, \tag{5.2}$$

where $B(X)$ denotes the unit ball of X.

Remark 5.1. A Banach space X with $R(X) < 2$ has the weak fixed point property, [26].

Let $1 < p < \infty$. A Banach space is said to have the Banach-Saks type p or property $(BS)_p$, if every weakly null sequence (x_k) has a subsequence (x_{kl}) such that for some $C > 0$,

$$\left\|\sum_{l=0}^{n} x_{kl}\right\| < C(n+1)^{1/p} \tag{5.3}$$

for all $n \in \mathbb{N}$ (see [27]).

Now, we may give the following results related to the some geometric properties, mentioned above, of the space $[\ell_p]_{e.r}$.

Theorem 5.1. *The space $[\ell_p]_{e.r}$ has the Banach-Saks type p.*

Proof. Let (ϵ_n) be a sequence of positive numbers for which $\sum \epsilon_n \le 1/2$, and also let (x_n) be a weakly null sequence in $B([\ell_p]_{e.r})$. Set $b_0 = x_0 = 0$ and $b_1 = x_{n_1} = x_1$. Then, there exists $m_1 \in \mathbb{N}$ such that

$$\left\|\sum_{i=m_1+1}^{\infty} b_1(i)e^{(i)}\right\|_{[\ell_p]_{e.r}} < \epsilon_1.$$

Since (x_n) is a weakly null sequence implies $x_n \to 0$ coordinatewise, there is an $n_2 \in \mathbb{N}$ such that

$$\left\|\sum_{i=0}^{m_2} x_n(i)e^{(i)}\right\|_{[\ell_p]_{e.r}} < \epsilon_1, \tag{5.4}$$

where $n \geq n_2$. Set $b_2 = x_{n_2}$. Then, there exists an $m_2 > m_1$ such that

$$\left\| \sum_{i=m_2+1}^{\infty} b_2(i)e^{(i)} \right\|_{[\ell_p]_{e.r}} < \epsilon_2. \tag{5.5}$$

By using the fact that $x_n \to 0$ coordinatewise, there exists an $n_3 > n_2$ such that

$$\left\| \sum_{i=0}^{m_2} x_n(i)e^{(i)} \right\|_{[\ell_p]_{e.r}} < \epsilon_2, \tag{5.6}$$

where $n \geq n_3$.

If we continue this process, we can find two increasing subsequences (m_i) and (n_i) such that

$$\left\| \sum_{i=0}^{m_j} x_n(i)e^{(i)} \right\|_{[\ell_p]_{e.r}} < \epsilon_j, \tag{5.7}$$

for each $n \geq n_{j+1}$ and

$$\left\| \sum_{i=m_1+1}^{\infty} b_1(i)e^{(i)} \right\|_{[\ell_p]_{e.r}} < \epsilon_1, \tag{5.8}$$

where $b_j = x_{n_j}$. Hence

$$\begin{aligned}
\left\| \sum_{j=0}^{n} b_j \right\|_{[\ell_p]_{e.r}} &= \left\| \sum_{j=0}^{n} \left(\sum_{i=0}^{m_{j-1}} b_j(i)e^{(i)} + \sum_{i=m_{j-1}+1}^{m_j} b_j(i)e^{(i)} + \sum_{i=m_j+1}^{\infty} b_j(i)e^{(i)} \right) \right\|_{[\ell_p]_{e.r}} \\
&= \left\| \sum_{j=0}^{n} \left(\sum_{i=0}^{m_{j-1}} b_j(i)e^{(i)} \right) \right\|_{[\ell_p]_{e.r}} + \left\| \sum_{j=0}^{n} \left(\sum_{i=m_{j-1}+1}^{m_j} b_j(i)e^{(i)} \right) \right\|_{[\ell_p]_{e.r}} \\
&\quad + \left\| \sum_{j=0}^{n} \left(\sum_{i=m_j+1}^{\infty} b_j(i)e^{(i)} \right) \right\|_{[\ell_p]_{e.r}} \\
&\leq \left\| \sum_{j=0}^{n} \left(\sum_{i=m_{j-1}+1}^{m_j} b_j(i)e^{(i)} \right) \right\|_{[\ell_p]_{e.r}} + 2\sum_{j=0}^{n} \epsilon_j.
\end{aligned}$$

On the other hand, it can be seen that $\|x_n\|_{[\ell_p]_{e.r}} < 1$. Therefore, $\|x_n\|^p_{[\ell_p]_{e.r}} < 1$. We have

$$\begin{aligned}&\left\|\sum_{j=0}^{n}\left(\sum_{i=m_{j-1}+1}^{m_j} b_j(i)e^{(i)}\right)\right\|^p_{[\ell_p]_{e.r}}\\ =&\sum_{j=0}^{n}\sum_{i=m_{j-1}+1}^{m_j}\left|\sum_{k=0}^{i}\frac{\binom{i}{k}q_k}{2^iQ_i}x_j(k)\right|^p\\ \leq&\sum_{j=0}^{n}\sum_{i=0}^{\infty}\left|\sum_{k=0}^{i}\frac{\binom{i}{k}q_k}{2^iQ_i}x_j(k)\right|^p\\ \leq& n+1.\end{aligned}$$

Hence, we obtain

$$\left\|\sum_{j=0}^{n}\left(\sum_{i=m_{j-1}+1}^{m_j} b_j(i)e^{(i)}\right)\right\|_{[\ell_p]_{e.r}} \leq (n+1)^{1/p}.$$

By using the fact $1 \leq (n+1)^{1/p}$ for all $n \in \mathbb{N}$, we have

$$\left\|\sum_{j=0}^{n} b_j\right\|_{[\ell_p]_{e.r}} \leq (n+1)^{1/p} + 1 \leq 2(n+1)^{1/p}.$$

Hence, $[\ell_p]_{e.r}$ has the Banach-Saks type p. This completes the proof of the theorem. □

Remark 5.2. Note that $R([\ell_p]_{e.r}) = R(\ell_p) = 2^{1/p}$, since $[\ell_p]_{e.r}$ is linearly isomorphic to ℓ_p.

Hence, by the Remarks 5.1 and 5.2, we have the following.

Theorem 5.2. *The space $[\ell_p]_{e.r}$ has the weak fixed point property, where $1 < p < \infty$.*

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