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Research article

The SIPG method of Dirichlet boundary optimal control problems with weakly imposed boundary conditions*

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Abstract: In this paper, we consider the symmetric interior penalty Galerkin (SIPG) method which is one of Discontinuous Galerkin Methods for the Dirichlet optimal control problems governed by linear advection-diffusion-reaction equation on a convex polygonal domain and the difficulties which we faced while solving this problem numerically. Since standard Galerkin methods have failed when the boundary layers have occurred and advection diffusion has dominated, these difficulties can occur in the cases of higher order elements and non smooth Dirichlet data in using standard finite elements. We find the most convenient natural setting of Dirichlet boundary control problem for the Laplacian and the advection diffusion reaction equations.After converting the continuous problem to an optimization problem, we solve it by "discretize-then-optimize" approach. In final, we estimate the optimal priori error estimates in suitable norms of the solutions and then support the result and the features of the method with numerical examples on the different kinds of domain.

Keywords: finite elements; discontinuous Galerkin methods; SIPG method; Dirichlet boundary; transposition method; optimal control; stabilized methods
Mathematics Subject Classification: 33F05, 65K10, 65M99, 65N30, 65N99

1. Introduction

Let Ω be a convex polygonal domain in $\mathbb{R}^2$. In this paper, we consider the following Dirichlet boundary optimal control problem,

$$\min_{\{q\}} J(y,q) = \frac{1}{2}\|y-\hat{y}\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|q\|^2_{L^2(\Gamma)} \tag{1.1}$$

subject to the advection-diffusion equation

$$-\Delta y(x) + \vec{\beta}(x)\cdot\nabla y(x) + c(x)y(x) = f(x), \qquad x \in \Omega, \tag{1.2a}$$

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$$y(x) = q(x), \qquad x \in \Gamma. \tag{1.2b}$$

Here, $y(x)$ denotes the state variable, $\hat{y}(x)$ is the desired state, (1.2a) and (1.2b) are called the state equation, $q(x)$ is the control, $\Gamma = \partial\Omega$.

We assume the given functions $f(x), \hat{y}(x) \in L^2(\Omega)$, $\vec{\beta}(x) \in [W^1_\infty(\Omega)]^2$, $c(x) \in L^\infty(\Omega)$ with the assumption

$$c(x) - \frac{1}{2}\nabla \cdot \vec{\beta}(x) \geq 0,$$

and $\alpha > 0$ is a given scalar.

This problem is important in many applications, for example distribution of pollution in air [1] or water [2] and for problems in computational electro-dynamics, gas and fluid dynamics [3]. However, there are several challenges involved in solving this problem numerically. One problem arises for higher order elements and nonsmooth Dirichlet data which can cause serious problems in using standard finite element methods (see [4, 5]). Another difficulty lies in the fact that Dirichlet boundary conditions do not enter the bilinear form naturally and that causes problems for analyzing the finite element method (see [6–10] for further discussion).

One faces another challenge in the presence of layers which are the regions where the gradient of the solution is large. Usually, the boundary layers occur because of the fact that problem has reduced to the first order PDEs and requires boundary conditions on inflow part of the boundary only. In this case, standard Galerkin methods fail when $h|\vec{\beta}| > 1$, where h is mesh size, producing highly oscillatory solutions. A lot of research has been done in last 40 years to address this difficulty (see [3, 4, 11–13]).

We have an example to illustrate this difficulty in the following simple example,

Example 1.1.

$$\begin{aligned} -\epsilon y''(x) + y'(x) &= 1, \quad x \in (0, 1), \\ y(0) = y(1) &= 0. \end{aligned} \tag{1.3}$$

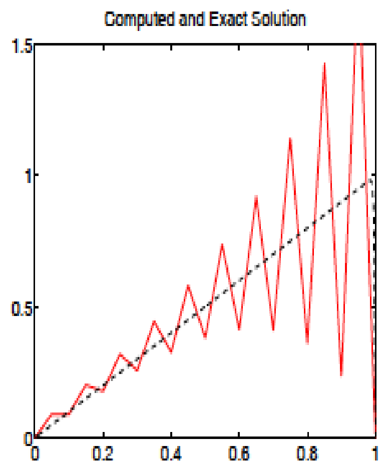

Figure 1. Standard Galerkin.

The Figure 1 *shows nonphysical oscillations of the standard Galerkin solution for* $h = 0.1$ *and* $\epsilon = 0.0025$.

One way to solve this problem is to use stabilized methods (see [14]). We will mention some of them. One of the first stable method of arbitrary order is SUPG (Streamline Upwind Petrov Galerkin) [11, 15, 16]. In this method, the space of test function is different from the space of trial function and chosen such that the method is stable and consistent. Other stabilized methods where the space of trial and test functions are the same and used upwind stabilization are HDG (Hybridizable Discontinuous Galerkin), [17–21], SIPG (Symmetric Interior Petrov Galerkin) [5, 7, 10], and LDG (Local Discontinuous Galerkin) [22–24]. Another popular stabilized method where the space of trial and test functions are the same is edge stabilization [25, 26].

DG methods are shown to be robust for the advection-diffusion-reaction problem (see [7, 27]) even for the advection-dominated case. DG methods were not only analyzed for the advection-diffusion-reaction problem but also for the optimal control problem of the advection-diffusion-reaction equation [28], (see other stabilized methods for the optimal control problem of the advection-diffusion-reaction equation [26, 29, 30]). In addition to being stable, the discontinuous Galerkin methods, such as SIPG, usually treat the boundary conditions weakly. The SIPG method was also analyzed for distributed optimal control problems and optimal local and global error estimates were obtained (see [28] but not for the boundary control problems. We would like to investigate the performance of the SIPG method applied to Dirichlet boundary control problem (1.1), ($1.2a$) and ($1.2b$) and prove a priori error estimates. We would also like to perform a number of numerical experiments to confirm our theoretical result which is the main subject of the current work.

In this paper, we analyze the SIPG solution of Dirichlet boundary control problem and the difficulties with dealing with the stability issues as well as with the difficulty of the treatment of Dirichlet boundary conditions. This method has some attractive features and offers some advantages. This method is stable and accurate, can be of arbitrary order and has been shown analytically that the boundary layers do not pollute the solution into the subdomain of smoothness [28]. Another attractive feature of the method is that Dirichlet boundary conditions are enforced weakly through the penalty term and not through the finite dimensional subspace [25]. As a result of the weak treatment of the boundary conditions, Dirichlet boundary control enters naturally into the bilinear form and makes analysis more natural [6–8, 31]. Finally, the SIPG method has the property that two strategies optimize-then-discretize and discretize-then-optimize produce the same discrete optimality system (see [10, 23]), which is not the case for other stabilized methods, for example, SUPG method (see [15]).

Let us show some features of SIPG method with Figure 2 in the previous example. Consider the problem 1.3 in the example 1 with the much more smaller diffiusion parameter 10^{-9} instead of -0.0025. Figure 2 shows the behavior of the SIPG solution for $h = 0.1$ and $\epsilon = 1e{-}9$. As one can see the solution is stable. The Dirichlet boundary condition at $x = 1$ is almost ignored by the method as a result of weak treatment.

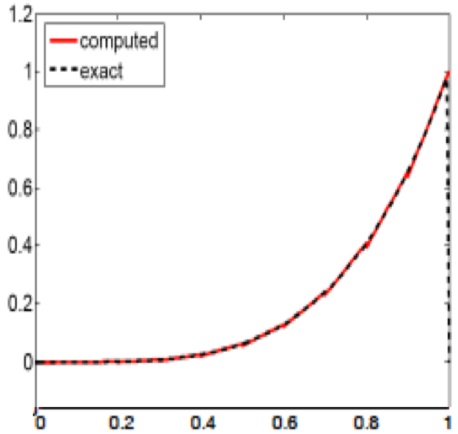

Figure 2. SIPG method.

Our choice of this particular DG method was motivated by good approximation and stabilization properties of the method. Additional attractive feature of the method is the weak treatment of the boundary conditions which allows us to set Dirichlet optimal control problem in natural the finite element frame work and to prove optimal convergence rates for on general convex polygonal domain. Moreover, we state the main result of the paper is valid for any general convex domain, there exists a positive constant C independent of h for the error between exact solution of the control function $\bar{q}$ and its approximation $\bar{q}_h$ such that

$$||\bar{q} - \bar{q}_h||_{L^2(\Gamma)} \leq Ch^{1/2}(|\bar{q}|_{H^{1/2}(\Gamma)} + ||\bar{y}||_{H^1(\Omega)} + ||\hat{y}||_{L^2(\Omega)}),$$

for h small enough. Also, we performed several numerical examples to support our theoretical results, and additionally when we investigate numerically performance of the method in the advection-dominated case.

2. Elliptic equations with Dirichlet boundary conditions

2.1. Preliminaries

Throughout the paper, we will use standard notation for spaces, completeness and norms. We will use the standard notation for Lebesgue and Sobolev space, their suitable norms, and L^2- inner product. Thus,

- $(u, v)_\Omega = \int_\Omega uvdx$ and $\langle u, v\rangle_\Gamma = \int_\Gamma uvds$ are the inner products on the domain Ω and its boundary Γ, respectively.
  The corresponding norms respectively are

$$||u||_{L^2(\Omega)} = (\int_\Omega |u|^2dx)^{1/2}, ||u||_{L^2(\Gamma)} = (\int_\Gamma |u|^2)ds)^{1/2}.$$

- $H^{1/2}(\Gamma) = \{u \in L^2(\Gamma) | \exists\tilde{u} \in H^1(\Omega) : u = tr(\tilde{u})\}$.

- $||u||_{H^{1/2}(\Gamma)} = inf\{||\tilde{u}||_{H^1(\Omega)}|tr(\tilde{u}) = u\}$.
- $|u|_{H^{1/2}(\Gamma)} = inf\{|\tilde{u}|_{H^1(\Omega)}|tr(\tilde{u}) = u\}$.

2.2. Setting the problem

First, let us consider the state equation,

$$\begin{aligned} -\Delta y + \vec{\beta} \cdot \nabla y + cy &= f \quad \text{in } \Omega, \\ y &= q \quad \text{on } \Gamma. \end{aligned} \tag{2.1}$$

We review some regularity results for various conditions on data which we will use later in the analysis. The first result is standard and found in [32].

Theorem 2.1. *Let $f \in H^{-1}(\Omega)$ and $q \in H^{1/2}(\Gamma)$. Then Eq* (2.1) *admits a unique solution $y \in H^1(\Omega)$. Moreover, the following estimate holds*

$$||y||_{H^1(\Omega)} \leq C\left(||f||_{H^{-1}(\Omega)} + ||q||_{H^{\frac{1}{2}}(\Gamma)}\right).$$

In the case of $q = 0$ on Γ, $f \in L^2(\Omega)$, and convex Ω, we can obtain a higher regularity of the solution (see [33]).

Theorem 2.2. *Let $f \in L^2(\Omega)$ and $q = 0$ on Γ. Then, the Eq* (2.1) *admits a unique solution $y \in H^2(\Omega)$ and the following estimate holds*

$$||y||_{H^2(\Omega)} \leq C||f||_{L^2(\Omega)}.$$

Remark 2.1. *Since the adjoint equation defined by*

$$\begin{aligned} -\Delta z - \nabla \cdot (\vec{\beta} z) + cz &= y - \hat{y} \quad \text{in } \Omega \\ z &= 0 \quad \text{on } \Gamma, \end{aligned}$$

it is also an advection-diffusion equation and the results of the above theorems are valid for the adjoint equation with similar estimates as well. Also, notice that $-\vec{\beta} \cdot \nabla z + (c - \nabla \cdot \vec{\beta})z = -\nabla \cdot (\vec{\beta} z) + cz$.

The theory in the case of $q \in L^2(\Gamma)$ is more technical and to obtain the desired regularity result, we use the transposition method [34], which we will briefly describe next.

2.3. The transposition method

Suppose q is smooth enough having continuous derivatives up to the desired order, $\phi \in L^2(\Omega)$ and let y_1 and y_2 be the solutions of the following equations,

$$\begin{aligned} -\Delta y_1 + \vec{\beta} \cdot \nabla y_1 + cy_1 &= 0 \quad \text{in } \Omega, \\ y_1 &= q \quad \text{on } \Gamma, \end{aligned} \qquad \text{and} \qquad \begin{aligned} -\Delta y_2 - \nabla \cdot (\vec{\beta} y_2) + cy_2 &= \phi \quad \text{in } \Omega, \\ y_2 &= 0 \quad \text{on } \Gamma, \end{aligned}$$

respectively. Then, by the integration by parts and using the fact that $y_2 = 0$ on Γ, we obtain

$$\begin{aligned}
0 &= (-\Delta y_1 + \vec{\beta} \cdot \nabla y_1 + c y_1, y_2)_\Omega \\
&= (\nabla y_1, \nabla y_2)_\Omega - \langle \frac{\partial y_1}{\partial n}, y_2 \rangle_\Gamma + \langle y_1 \vec{\beta} \cdot \vec{n}, y_2 \rangle_\Gamma - (y_1, \nabla \cdot (\vec{\beta} y_2))_\Omega + (c y_1, y_2)_\Omega \\
&= (\nabla y_1, \nabla y_2)_\Omega - (y_1, \nabla \cdot (\vec{\beta} y_2))_\Omega + (y_1, c y_2)_\Omega \\
&= (y_1, -\Delta y_2)_\Omega + \langle y_1, \frac{\partial y_2}{\partial n} \rangle_\Gamma - (y_1, \nabla \cdot (\vec{\beta} y_2))_\Omega + (y_1, c y_2)_\Omega \\
&= (y_1, -\Delta y_2 - \nabla \cdot (\vec{\beta} y_2) + c y_2)_\Omega + \langle y_1, \frac{\partial y_2}{\partial n} \rangle_\Gamma \\
&= (y_1, \phi)_\Omega + \langle q, \frac{\partial y_2}{\partial n} \rangle_\Gamma,
\end{aligned}$$

where in the last step we use that $-\Delta y_2 - \nabla \cdot (\vec{\beta} y_2) + c y_2 = \phi$ in Ω and $y_1 = q$ on Γ. Hence we obtain

$$(y_1, \phi)_\Omega = -\langle q, \frac{\partial y_2}{\partial n} \rangle_\Gamma.$$

The above formula defines a mapping $\Lambda : \phi \to -\frac{\partial y_2}{\partial n}$ that is linear and continuous from $L^2(\Omega)$ to $H^{1/2}(\Gamma)$. Since the embedding $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$ is compact, Λ is a compact operator from $L^2(\Omega)$ to $L^2(\Gamma)$. Hence, its adjoint Λ^* is a compact operator from $L^2(\Gamma)$ to $L^2(\Omega)$.

Since $(y_1, \phi)_\Omega = -\int_\Gamma q \frac{\partial y_2}{\partial n} = \langle q, \Lambda \phi \rangle_\Gamma$ and $\langle q, \Lambda \phi \rangle_\Gamma = (\Lambda^* q, \phi)_\Omega$, we conclude that $y_1 = \Lambda^* q$. Using the above, we can define an "ultra-weak" solution for the Eq (2.1) for Dirichlet data in $L^2(\Gamma)$ as follows.

Definition 2.1. *We say that $y \in L^2(\Omega)$ is a unique ultra-weak solution of the Eq* (2.1) *if*

$$\int_\Omega y\phi = (f, p)_{(H^{-1}(\Omega), H_0^1(\Omega))} - \int_\Gamma q \frac{\partial p}{\partial n}, \quad \forall \phi \in L^2(\Omega),$$

where p satisfies

$$\begin{aligned}
-\Delta p - \nabla \cdot (\vec{\beta} p) + c p &= \phi \quad \text{in } \Omega, \\
p &= 0 \quad \text{on } \Gamma.
\end{aligned}$$

Now we are ready to provide the following regularity result.

Theorem 2.3. *For any $f \in H^{-1}(\Omega)$ and $q \in L^2(\Gamma)$, the Eq* (2.1) *admits a unique ultra-weak solution $y \in L^2(\Omega)$. Moreover, the following estimate holds,*

$$\|y\|_{L^2(\Omega)} \leq C(\|f\|_{H^{-1}(\Omega)} + \|q\|_{L^2(\Gamma)}). \tag{2.2}$$

Proof. Existence follows from the Definition (2.1). For the uniqueness, we assume that y_1 and y_2 are distinct solutions of the Eq (2.1) and let $u = y_1 - y_2$, then

$$\begin{aligned}
-\Delta u - \nabla \cdot (\vec{\beta} u) + c u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \Gamma.
\end{aligned}$$

Since $H^1(\Omega)$ is dense in $L^2(\Omega)$, it is enough to consider $u \in H^1(\Omega)$. By the Theorem (2.1), we have

$$\|u\|_{H^1(\Omega)} = 0.$$

As a result $u = 0$, hence $y_1 = y_2$ and this contradiction proves the uniqueness.

To show the desired estimate (2.2), we use a duality argument. Let w be the solution of the problem

$$\begin{aligned} -\Delta w - \nabla \cdot (\vec{\beta} w) + cw = y \quad & \text{in } \Omega, \\ w = 0 \quad & \text{on } \Gamma. \end{aligned}$$

By using the above duality argument and using integration by parts and the fact that $w = 0$ on Γ, we obtain

$$\begin{aligned} \|y\|^2_{L^2(\Omega)} &= (y, -\Delta w - \nabla \cdot (\vec{\beta} w) + cw)_\Omega \\ &= (\nabla y, \nabla w)_\Omega - \langle y, \frac{\partial w}{\partial n} \rangle_\Gamma - \langle y, w(\vec{\beta} \cdot \vec{n}) \rangle_\Gamma + (\vec{\beta} \cdot \nabla y, w)_\Omega + (y, cw)_\Omega \\ &= (-\Delta y, w)_\Omega + \langle \frac{\partial y}{\partial n}, w \rangle_\Gamma - \langle y, \frac{\partial w}{\partial n} \rangle_\Gamma - \langle y, w(\vec{\beta} \cdot \vec{n}) \rangle_\Gamma + (\vec{\beta} \cdot \nabla y, w)_\Omega + (y, cw)_\Omega \\ &= (-\Delta y + \vec{\beta} \cdot \nabla y + cy, w)_\Omega - \langle y, \frac{\partial w}{\partial n} \rangle_\Gamma \\ &= (f, w)_\Omega - \langle q, \frac{\partial w}{\partial n} \rangle_\Gamma, \end{aligned}$$

where in the last step we use $-\Delta y + \vec{\beta} \cdot \nabla y + cy = f$.

By the trace and the Cauchy-Schwarz inequalities, and by using the Theorem (2.2), we have the following estimate

$$\begin{aligned} \|y\|^2_{L^2(\Omega)} &\le \|f\|_{H^{-1}(\Omega)} \|w\|_{H^1(\Omega)} + \|q\|_{L^2(\Gamma)} \|\frac{\partial w}{\partial n}\|_{L^2(\Gamma)} \\ &\le C(\|f\|_{H^{-1}(\Omega)} + \|q\|_{L^2(\Gamma)}) \|w\|_{H^2(\Omega)} \\ &\le C(\|f\|_{H^{-1}(\Omega)} + \|q\|_{L^2(\Gamma)}) \|y\|_{L^2(\Omega)}. \end{aligned}$$

Canceling $\|y\|_{L^2(\Omega)}$ on both sides, we prove the desired estimate (2.2). □

3. First order optimality system and the regularity of the optimal solution

Next we will provide the first order optimality conditions for the problem (1.1)

Theorem 3.1. *Assume that $f, \hat{y} \in L^2(\Omega)$ and let $(\bar{y}, \bar{q})$ be the optimal solution of the Eq* (2.1). *Then, the optimal control $\bar{q}$ is given by $\frac{\partial \bar{z}}{\partial n} = \alpha \bar{q}$, where $\bar{z}$ is the unique solution of the equation,*

$$\begin{aligned} -\Delta \bar{z} - \nabla \cdot (\vec{\beta} \bar{z}) + c\bar{z} = \bar{y} - \hat{y} \quad & \text{in } \Omega, \\ \bar{z} = 0 \quad & \text{on } \Gamma. \end{aligned} \tag{3.1}$$

Proof. Let $(\bar{y}, \bar{q})$ be an optimal solution of the Eq (1.1). We set

$$F(q) = J(y(q), q),$$

where $y(q)$ is the solution of the Eq (2.1) for a given $q \in L^2(\Gamma)$. Let y_q be the solution of the problem

$$\begin{aligned} -\Delta y_q + \vec{\beta} \cdot \nabla y_q + c y_q &= f \qquad \text{in } \Omega, \\ y_q &= q + \bar{q} \quad \text{on } \Gamma. \end{aligned}$$

By the optimality of $(\bar{y}, \bar{q})$ and convexity of Ω, we have that $\frac{1}{\lambda}(F(\bar{q} + \lambda q) - F(\bar{q})) \geq 0$ for all q and $\lambda \in (0, 1]$ [35]. For $\lambda = 1$, $y_q = q + \bar{q}$, and so $F(\bar{q} + q) - F(\bar{q}) \geq 0$.

Equivalently, if $F(\bar{q} + q) - F(\bar{q}) \geq 0$ for all q in $L^2(\Gamma)$, then $\bar{q}$ is an optimal solution of the problem. We find

$$\begin{aligned} F(\bar{q} + q) - F(\bar{q}) &= J(y_q, q + \bar{q}) - J(\bar{y}, \bar{q}) \\ &= \frac{1}{2} \int_\Omega (y_q - \bar{y})(y_q + \bar{y} - 2\hat{y}) + \frac{\alpha}{2} \int_\Gamma (2q\bar{q} + q^2) \\ &= \frac{1}{2} \int_\Omega (y_q - \bar{y})^2 + \frac{\alpha}{2} \int_\Gamma q^2 + \int_\Omega (y_q - \bar{y})(\bar{y} - \hat{y}) + \alpha \int_\Gamma q\bar{q}. \end{aligned}$$

Let $\bar{z}$ be the solution of the Eq (3.1). Then, we can estimate the third term of the right hand side by using the Green's formula and using the fact that $y_q = \bar{q} + q$ and $\bar{z} = 0$ on Γ. Thus, we obtain

$$\begin{aligned} &\int_\Omega (y_q - \bar{y})(\bar{y} - \hat{y}) = \int_\Omega (y_q - \bar{y})(-\Delta \bar{z} - \nabla \cdot (\vec{\beta}\bar{z}) + c\bar{z}) \\ &= -\int_\Gamma \frac{\partial \bar{z}}{\partial n}(y_q - \bar{q}) + \int_\Omega \nabla \bar{z} \cdot \nabla (y_q - \bar{y}) - \int_\Gamma (y_q - \bar{y})\bar{z}(\vec{\beta} \cdot \vec{n}) + \int_\Omega \bar{z}(\vec{\beta} \cdot \nabla (y_q - \bar{y})) + \int_\Omega (y_q - \bar{y}) c\bar{z} \\ &= -\int_\Gamma \frac{\partial \bar{z}}{\partial n}(\bar{q} + q - \bar{q}) + \int_\Omega \nabla \bar{z} \cdot \nabla (y_q - \bar{y}) + \int_\Omega \bar{z}(\vec{\beta} \cdot \nabla (y_q - \bar{y})) + \int_\Omega (y_q - \bar{y}) c\bar{z} \\ &= -\int_\Gamma q\frac{\partial \bar{z}}{\partial n} + \left(\frac{\partial y_q}{\partial n} - \frac{\partial \bar{y}}{\partial n}\right)\bar{z}\Big|_\Gamma + \int_\Omega \bar{z}\,\underbrace{(-\Delta(y_q - \bar{y}) + \vec{\beta} \cdot \nabla (y_q - \bar{y}) + c(y_q - \bar{y}))}_{=0}. \end{aligned}$$

Notice that $\int_\Omega \nabla \bar{z} \cdot \nabla (y_q - \bar{y}) = (\frac{\partial y_q}{\partial n} - \frac{\partial \bar{y}}{\partial n})\bar{z}\Big|_\Gamma - \int_\Omega \bar{z}\Delta(y_q - \bar{y})$ by using integration by parts. By setting $\frac{\partial \bar{z}}{\partial n} = \alpha\bar{q}$, we have

$$\int_\Omega (y_q - \bar{y})(\bar{y} - \hat{y}) = -\int_\Gamma q\frac{\partial \bar{z}}{\partial n} = -\alpha \int_\Gamma q\bar{q}.$$

Putting all results together, we have

$$\begin{aligned} F(\bar{q} + q) - F(\bar{q}) &= \frac{1}{2} \int_\Omega (y_q - \bar{y})^2 + \frac{\alpha}{2} \int_\Gamma q^2 - \alpha \int_\Gamma q\bar{q} + \alpha \int_\Gamma q\bar{q} \\ &= \frac{1}{2} \int_\Omega (y_q - \bar{y})^2 + \frac{\alpha}{2} \int_\Gamma q^2 \geq 0, \end{aligned}$$

i.e., $(\bar{y}, \bar{q})$ is the optimal solution to the Eq (2.1) with $\bar{q} = \frac{1}{\alpha}\frac{\partial \bar{z}}{\partial n}$ where $\alpha > 0$ given any scalar in the problem (1.1) □

3.1. Strong form of the first order optimality conditions

The first order optimality conditions in the strong form are as the following

$$Adjoint\ equation \begin{cases} -\Delta z - \vec{\beta}\cdot\nabla z + (c - \nabla\cdot\vec{\beta})z = y - \hat{y} & in \quad \Omega, \\ z = 0 & on \quad \Gamma. \end{cases} \tag{3.2}$$

$$Gradient\ equation \left\{ \frac{\partial z}{\partial n} = \alpha q \quad on \quad \Gamma, \right. \tag{3.3}$$

$$State\ equation \begin{cases} -\Delta y + \vec{\beta}\cdot\nabla y + cy = f & in \quad \Omega, \\ y = q & on \quad \Gamma. \end{cases} \tag{3.4}$$

3.2. Regularity

In the next theorem, we establish the regularity of the optimal solution of the problem (1.2a) and (1.2b).

Theorem 3.2. *Let* $(\bar{y}, \bar{q}) \in L^2(\Omega) \times L^2(\Gamma)$ *be the optimal solution to the optimization problem* (1.1) *subject to the problem* (1.2a) *and* (1.2b)*, and* $\bar{z}$ *be the optimal adjoint state* (3.1)*. Then,*

$$(\bar{y}, \bar{q}, \bar{z}) \in H^1(\Omega) \times H^{1/2}(\Gamma) \times H^2(\Omega).$$

Proof. For $\bar{q} \in L^2(\Gamma)$, from the state Eq (3.4), $\bar{y} \in L^2(\Omega)$ holds by Theorem (2.3).

Since $\bar{y}, \hat{y} \in L^2(\Omega)$ and Ω is a convex domain, from the adjoint Eq (3.2), $\bar{z} \in H^2(\Omega)$ holds by Theorem (2.2).

Since $\bar{z} \in H^2(\Omega)$, we have $\frac{\partial \bar{z}}{\partial n} \in H^{1/2}(\Gamma)$, from the gradient Eq (3.3), $\frac{\partial \bar{z}}{\partial n} = \alpha q$ implies $\bar{q} \in H^{1/2}(\Gamma)$.

Since $\bar{q} \in H^{1/2}(\Gamma)$, from the state Eq (3.4), $\bar{y} \in H^1(\Omega)$ holds by Theorem (2.1). □

Remark 3.1. *Using regularity results, we can generalize the regularity which depends on the largest interior angle of the polygonal domain in* $\mathbb{R}^2$ *[36].*

4. Discontinuous Galerkin discretization

The idea of the FEM is to construct V_h and Q_h defined on a finite dimensional space that is well approximate the solution spaces V and Q. The Galerkin FEM is to find $y_h \in V_h$ and $q_h \in Q_h$ such that

$$a_h(y_h, v_h) = \ell_h(f; q_h, v_h), \qquad \forall v_h \in V_h, \tag{4.1}$$

where V_h is a finite dimensional space and h is a discretization parameter. We can easily see that if $a_h(\cdot,\cdot)$ satisfies the conditions of *Lax-Milgram* Lemma, the Eq (4.1) has a unique solution for each h.

To construct V_h, we consider a family of conforming quasi-uniform shape regular triangulations T_h of Ω such that $\bar{\Omega} = \cup_{\tau_i \in T_h} \tau_i$ and $\tau_i \cap \tau_j = 0\ \forall \tau_i, \tau_j \in T_h,\ i \neq j$ with a mesh size

$$h = sup_{\tau_i \in T_h} diam(\tau_i).$$

We define E_h as a collection of all edges $E_h = E_h^0 \cup E_h^\partial$ where E_h^0 and E_h^∂ are the collections of interior and boundary edges, respectively, and we decompose the boundary edges as

$$E_h^\partial = E_{h\,h}^{+\,-}, \tag{4.2}$$

where $E_h^- := \{e \in E_h^\partial : e \subset \{x \in \Gamma : \vec{\beta}(x) \cdot \vec{n}(x) < 0\}\}$ and $E_h^+ := E_h^\partial \backslash E_h^-$ i.e. these are the collections of the edges that belong to the inflow and outflow part of the boundary, respectively. In other words, for a given elements $\tau \in T_h$ and n_τ indicates the outward normal to τ, then we can decompose its boundary $\partial\tau$ as $\partial\tau_- = \{x \in \partial\tau : \vec{\beta}(x) \cdot \vec{n_\tau}(x) < 0\}$ and $\partial\tau_+ = \{x \in \partial\tau : \vec{\beta}(x) \cdot \vec{n_\tau}(x) \geq 0\}$.

We define the standard jumps and averages on the set of interior edges by

$$\{\varphi\} = \frac{\varphi_1 + \varphi_2}{2}, \qquad [[\varphi]] = \varphi_1 \vec{n}_1 + \varphi_2 \vec{n}_2,$$
$$\{\vec{\phi}\} = \frac{\vec{\phi}_1 + \vec{\phi}_2}{2}, \qquad [[\vec{\phi}]] = \vec{\phi}_1 \cdot \vec{n}_1 + \vec{\phi}_2 \cdot \vec{n}_2,$$

where $\vec{n}_1$ and $\vec{n}_2$ are outward normal vectors at the common boundary edge of neighboring elements τ_1 and τ_2, respectively. If $e \in E_h^\partial$, then $\{\varphi\} = [[\varphi]] = \varphi|_e$ [37, 38]. Define the discrete state and control spaces as

$$V_h := \{y_h \in L^2(\Omega) : y_h|_\tau \in \mathcal{P}_k(\tau) \quad \forall \tau \in T_h\}, \tag{4.3}$$

$$Q_h := \{q_h \in L^2(\Gamma) : q_h|_\tau \in \mathcal{P}_l(\tau) \quad \forall \tau \in E_h^\partial\}, \tag{4.4}$$

respectively. We denote by $\mathcal{P}_k$,$\mathcal{P}_l$ the space of polynomials of degree at most k on each element and at most l on each edge, respectively. In general, the state and control variables can be approximated by polynomials of different degrees k,$l \in \mathbb{N}$.

Here, we use the symmetric interior penalty Galerkin (SIPG) method to approximate to the problem. In deriving the SIPG method, we use the following identity

$$\begin{aligned}\sum_{\tau \in T_h} (\vec{\phi} \cdot \vec{n}, \varphi)_{\partial\tau} &= \sum_{e \in E_h} (\{\vec{\phi}\}, [[\varphi]])_e + \sum_{e \in E_h^0} ([[\vec{\phi}]], \{\varphi\})_e \\ &= \sum_{e \in E_h^0} (\{\vec{\phi}\}, [[\varphi]])_e + ([[\vec{\phi}]], \{\varphi\})_e + \sum_{e \in E_h^\partial} (\vec{\phi} \cdot \vec{n}, \varphi)_e.\end{aligned}$$

The SIPG solutions $q_h \in Q_h$, $y_h \in V_h$ and a constant advection field $\vec{\beta}$ satisfies the Eq (4.1) for all $v_h \in V_h$ where

$$\begin{aligned}a_h(y_h, v_h) = &\sum_{\tau \in T_h} (\nabla y_h, \nabla v_h)_\tau + \sum_{\tau \in T_h} (\vec{\beta} \cdot \nabla y_h + c y_h, v_h)_\tau \\ &+ \sum_{e \in E_h} [\frac{\gamma}{h}([[y_h]], [[v_h]])_e - (\{\nabla y_h\}, [[v_h]])_e - ([[y_h]], \{\nabla v_h\})_e] \\ &+ \sum_{e \in E_h^0} (y_h^+ - y_h^-, |\vec{n} \cdot \vec{\beta}| v_h^+)_e + \sum_{e \in E_h^-} (y_h^+, v_h^+ |\vec{n} \cdot \vec{\beta}|)_e,\end{aligned} \tag{4.5}$$

where γ is the penalty parameter, which should be chosen sufficiently large to ensure the stability of the SIPG scheme [37, 39, 40], and $y_h^- = \lim_{\zeta \to 0^+} y_h(x - \zeta\vec{\beta})$, $y_h^+(x) = \lim_{\zeta \to 0^+} y_h(x + \zeta\vec{\beta})$,

$$\ell_h(f; q_h, v_h) = \sum_{\tau \in T_h} (f, v_h)_\tau + \sum_{e \in E_h^\partial} (\frac{\gamma}{h}(q_h, [[v_h]])_e - (q_h, \{\nabla v_h\})_e) + \sum_{e \in E_h^-} (q_h, v_h^+ |\vec{n} \cdot \vec{\beta}|)_e. \tag{4.6}$$

Then, DG solution is defined as a solution of $a_h(y_h, v_h) = \ell_h(f; q_h, v_h)$ for al all $v_h \in V_h$, and mesh dependent norm

$$|||v_h|||^2 = ||v_h||_h^2 = \sum_{\tau\in T_h} ||\nabla v_h||_\tau^2 + ||v_h||_\tau^2 + \sum_{e\in E_h^\partial} \frac{\gamma}{h} ||[[v_h]]||_e^2,$$

which is equivalent to the energy norm [38].

4.1. Well-posed

It has been shown, for example [7], that the bilinear form (4.5) is coercive and bounded on V_h i.e., $a_h(v_h, v_h) \geq C|||v_h|||^2$ and $a_h(y_h, v_h) \leq C|||y_h||| |||v_h|||$, respectively. Thus, *Lax-Milgram Lemma* guarantees the existence of a unique solution $y_h \in V_h$ of the Eq (4.1) for all $v_h \in V_h$.

4.2. Discrete optimality system

We apply the SIPG discretization to the optimal control problem (1.1). Now, define the discrete Lagrangian as

$$L_h(\bar{y}_h, \bar{q}_h, \bar{z}_h) = J(\bar{y}_h, \bar{q}_h) + a_h(\bar{y}_h, \bar{z}_h) - \ell_h(f, \bar{q}_h).$$

Then, setting the partial Frechet derivatives with respect to y_h, q_h and z_h to be zero, we obtain the discrete optimality system.Then, the discretized optimal control problem has a unique solution $(y_h, q_h) \in V_h x Q_h$ if only if there exists $z_h \in V_h$ holds the following system:

$$\frac{\partial L_h}{\partial \bar{y}_h}\psi_h = 0 \Rightarrow a_h(\psi_h, \bar{z}_h) = \ell_h(\hat{y} - \bar{y}_h; 0, \psi_h) \quad \forall \psi_h \in V_h, \tag{4.7}$$

$$\frac{\partial L_h}{\partial \bar{q}_h}\phi_h = 0 \Rightarrow \langle \frac{\partial \bar{z}_h}{\partial n}, \phi_h\rangle_\Gamma = -\langle \alpha \bar{q}_h, \phi_h\rangle_\Gamma + \frac{\gamma}{h}\langle \bar{z}_h, \phi_h\rangle_\Gamma + \langle \bar{z}_h|\vec{n}\cdot\vec{\beta}|, \phi_h\rangle_\Gamma \;\; \forall \phi_h \in Q_h, \tag{4.8}$$

$$\frac{\partial L_h}{\partial \bar{z}_h}\varphi_h = 0 \Rightarrow a_h(\varphi_h, \bar{y}_h) = \ell_h(f; q_h, \varphi_h) \quad \forall \varphi_h \in V_h. \tag{4.9}$$

5. DG error estimates

5.1. Auxiliary estimates

We will need some auxiliary estimates that we will use in the proof of the main result. First, we have some standard estimates which are trace and inverse inequalities and the proofs can be found in [41–43].

Lemma 5.1. *There exist positive constants C_{tr} and C_{inv} independent of τ and h such that for $\forall \tau \in T_h$,*

$$||v||_{\partial\tau} \leq C_{tr}(h^{-1/2}||v||_\tau + h^{1/2}||\nabla v||_\tau), \quad \forall v \in H^{k+1}(\tau), \tag{5.1}$$

$$||\nabla v_h||_\tau \leq C_{inv} h^{-1} ||v_h||_\tau, \qquad \forall v_h \in V_h, \tag{5.2}$$

for integer $k \geq 0$

Then, we need some basic estimates for L^2-Projection where $P_h : L^2(\Omega) \to V_h$ is the orthogonal projection such that $(P_h v, \chi)_\tau = (v, \chi)_\tau$ for all $v \in L^2(\tau)$ and $\chi \in V_h$.

Lemma 5.2. *Let P_h be L^2-projection. Then, we have that $\exists P_h : H^{k+1} \to V_h$ such that for any $\tau \in T_h$,*

$$\|v - P_h v\|_{L^2(\tau)} \le Ch^{k+1}\|v\|_{H^{k+1}(\tau)}, \quad \forall v \in H^{k+1}(\tau),$$
$$\|\nabla(v - P_h v)\|_{L^2(\tau)} \le Ch^{k}\|v\|_{H^{k+1}(\tau)}, \quad \forall v \in H^{k+1}(\tau),$$

where integer $k \ge 0$.

Proof. From *Local Approximation* used in [30], we know that there exists a local interpolant operator $P_h : H^{k+1} \to V_h$ such that for any $\tau \in T_h$ and $\forall v \in H^{k+1}(\tau)$,

$$h\|\nabla(v - P_h(v))\|_\tau + \|v - P_h(v)\|_\tau \le Ch^{k+1}\|v\|_{H^{k+1}(\tau)}.$$

Since $h\|\nabla(v - P_h(v))\|_\tau \le h\|\nabla(v - P_h(v))\|_\tau + \|v - P_h(v)\|_\tau \le Ch^{k+1}\|v\|_{H^{k+1}(\tau)}$, we have

$$h\|\nabla(v - P_h(v))\|_\tau \le Ch^{k+1}\|v\|_{H^{k+1}(\tau)}.$$

Thus,

$$\|\nabla(v - P_h v)\|_{L^2(\tau)} \le Ch^{k}\|v\|_{H^{k+1}(\tau)}.$$

Likewise, we obtain $\|v - P_h v\|_{L^2(\tau)} \le Ch^{k+1}\|v\|_{H^{k+1}(\tau)}$. □

Now, we are ready to show the error estimate of SIPG solution in the energy norm.

Lemma 5.3. *Let v be the unique solution of the Eq* (2.1) *to satisfy $v \in H^{k+1}(\Omega)$ and $v_h \in V_h$ be the SIPG solution of the discretized state equation with piecewise polynomials of degree k. Then,*

$$|||v - v_h||| \le Ch^{k}\|v\|_{H^{k+1}(\Omega)},$$

for integer $k \ge 0$.

The proof can be easily seen by using the well-posedness of the bilinear form (4.5), Lemmas (5.1) and (5.2), and it can be also found for example in [44, 45]. Next, we will need the estimate of L^2-Projection on the boundary Γ where $P_h^\partial : L^2(\Gamma) \to Q_h$ is defined by $\langle q - P_h^\partial q, \phi_h\rangle_e = 0$ for all $\phi_h \in \mathcal{P}_s(e)$.

Lemma 5.4. *Let P_h^∂ be L^2−projection defined on the boundary. Then, for any edge $e \in E_h^\partial$,*

$$\|q - P^\partial q\|_{L^2(e)} + h^s\|q - P^\partial q\|_{W^{s,p}(e)} \le h^s|q|_{W^{s,p}(e)} \quad \forall e \in E_h^\partial,$$

where E_h^∂ is the set of boundary edges which is described in the Eq (4.2)*, $q \in W^{s,p}(e)$, $0 \le s \le 1$, and $1 < p < \infty$.*

The proof can be found in [6].

Note that Lemmas (5.1)–(5.4) state for general reqularity which depends on the polynomial degree k used in SIPG, the regularity on the solution of the optimal control problem is $(\bar{y}, \bar{q}, \bar{z}) \in H^1(\Omega) \times H^{1/2}(\Gamma) \times H^2(\Omega)$ by Theorem (3.2). Thus, the following estimates and the main result will be done by using the regularity on the solution of the problem in Theorem (3.2).

Since SIPG method treats the boundary conditions weakly, SIPG solution is not zero on the boundary even if its continuous solution z is. However, the following result says that the norm of SIPG solution z_h on the boundary is rather small.

Lemma 5.5. *Let us define auxiliary variable $\tilde{z}$ to be a solution of the Eq* (3.2)

$$\begin{aligned} -\Delta\tilde{z} - \nabla\cdot(\vec{\beta}\tilde{z}) + c\tilde{z} &= \hat{y} - y_h \ \ in\ \Omega \\ \tilde{z} &= 0 \qquad on\ \Gamma, \end{aligned}$$

and $\tilde{z}_h \in V_h$ be the SIPG approximation solution. Then,

$$||\tilde{z}_h||_{L^2(\Gamma)} \leq Ch^{3/2}||\hat{y} - y_h||_{L^2(\Omega)}.$$

Proof. Let $\tilde{z}$ be a solution to the Eq (3.2). Since

$$||\tilde{z}_h||_{L^2(\Gamma)} = ||\tilde{z}_h - \tilde{z}||_{L^2(\Gamma)} = ||[[\tilde{z}_h - \tilde{z}]]||_{L^2(\Gamma)},$$

we can estimated that

$$||[[\tilde{z}_h - \tilde{z}]]||_{L^2(\Gamma)} \leq Ch^{1/2}|||\tilde{z}_h - \tilde{z}|||,$$

by using the definition of the energy norm. Thus, by Theorems (2.2), (3.2) and Lemma (5.3), we have that

$$||\tilde{z}_h||_{L^2(\Gamma)} \leq Ch^{1/2}|||\tilde{z}_h - \tilde{z}||| \leq Ch^{1/2}h||\tilde{z}||_{H^2(\Omega)} \leq Ch^{3/2}||\hat{y} - y_h||_{L^2(\Omega)}.$$

□

The estimate of $|||y - y_h|||$ is more involved because $(y-y_h)$ does not satisfy the Galerkin orthogonality by $(y - y_h) \notin V_h$ and $a_h(y - y_h, v_h) \neq 0$ for $v_h \in V_h$. First, we can show the following result.

Lemma 5.6. *Let y and y_h satisfy*

$$\begin{aligned} a_h(y, v) &= \ell_h(f; q, v), \qquad \forall v \in H^1(\Omega), \\ a_h(y_h, \chi) &= \ell_h(f; q_h, \chi), \qquad \forall \chi \in V_h. \end{aligned}$$

Then,

$$|||y - y_h||| \leq C(h^{-1/2}||q - q_h||_{L^2(\Gamma)} + ||y||_{H^1(\Omega)}).$$

Proof. By the coersivity, adding and subtracting $P_h y$, we have

$$|||y - y_h|||^2 \leq a_h(y - y_h, y - y_h) = \underbrace{a_h(y - y_h, P_h y - y_h)}_{I} + \underbrace{a_h(y - y_h, y - P_h y)}_{II}.$$

II :
By using the boundedness of $a_h(.,.)$, Theorem (3.2) and Lemma (5.3), $k = 0$ and we obtain

$$a_h(y - y_h, y - P_h y) \leq |||y - y_h||||||y - P_h y||| \leq C|||y - y_h|||||y||_{H^1(\Omega)}.$$

I :
Since$(P_h y - y_h) \in V_h$, we have $a_h(y - y_h, P_h y - y_h) = \ell_h(0; q - q_h, P_h y - y_h)$. Then, we have

$$\begin{aligned} \ell_h(0; q - q_h, P_h y - y_h) = &\sum_{e \in E_h^\partial} (\frac{\gamma}{h}(q - q_h, [[P_h y - y_h]])_e - (q - q_h, \{\nabla(P_h y - y_h)\})_e) \\ &+ \sum_{e \in E_h^-} (q - q_h, (P_h y - y_h)^+ |\vec{n}\cdot\vec{\beta}|)_e. \end{aligned}$$

By the definition of $\ell_h(,)$, we can see that $\sum_e \frac{\gamma}{h}(q-q_h, [[P_h y-y_h]])_e$ is the dominating term by being $\frac{\gamma}{h}$ large. Using the fact that $\|[[P_h y-y_h]]\|_{L^2(\Gamma)}$ is a part of the energy norm and Lemma (5.3) for $k=0$ since $y \in H^1(\Omega)$, we have

$$\begin{aligned}
&\ell_h(0; q-q_h, P_h y-y_h) \le C\sum_e \frac{\gamma}{h}(q-q_h, [[P_h y-y_h]])_e \\
&\le Ch^{-1}\left(\sum_e \|q-q_h\|^2_{L^2(e)}\right)^{1/2}\left(\sum_e \|[[P_h y-y_h]]\|^2_{L^2(e)}\right)^{1/2} \\
&\le Ch^{-1}\|q-q_h\|_{L^2(\Gamma)}\|[[P_h y-y_h]]\|_{L^2(\Gamma)} \le Ch^{-1}\|q-q_h\|_{L^2(\Gamma)}h^{1/2}|||P_h y-y_h||| \\
&\le Ch^{-1/2}\|q-q_h\|_{L^2(\Gamma)}(|||P_h y-y||| + |||y-y_h|||) \\
&\le Ch^{-1/2}\|q-q_h\|_{L^2(\Gamma)}(\|y\|_{H^1(\Omega)} + |||y-y_h|||).
\end{aligned}$$

The other terms in $\ell_h(0; q-q_h, P_h y-y_h)$ can be estimated with the similar way. Thus,

$$\begin{aligned}
&|||y-y_h|||^2 \le I + II \\
&\le C\|y\|_{H^1(\Omega)}|||y-y_h||| + Ch^{-\frac{1}{2}}\|q-q_h\|_{L^2(\Gamma)}|||y-y_h||| + Ch^{-\frac{1}{2}}\|q-q_h\|_{L^2(\Gamma)}\|y\|_{H^1(\Omega)} \\
&\le \frac{1}{4}|||y-y_h|||^2 + Ch^{-1}\|q-q_h\|^2_{L^2(\Gamma)} + C\|y\|^2_{H^1(\Omega)}.
\end{aligned}$$

By first taking the square root and then canceling $|||y-y_h|||$, we obtain

$$|||y-y_h||| \le C(h^{-1/2}\|q-q_h\|_{L^2(\Gamma)} + \|y\|_{H^1(\Omega)}).$$

□

Using a duality, we can show better estimate in L^2 norm.

Lemma 5.7. *Let y be the solution of the Eq* (2.1) *and y_h in V_h satisfy the bilinear form* (4.5)*. Then,*

$$\|y-y_h\|_{L^2(\Omega)} \le C(h^{1/2}\|q-q_h\|_{L^2(\Gamma)} + h\|y\|_{H^1(\Omega)}).$$

Proof. Since y_h is not a Galerkin projection of y, let us define $\tilde{y}_h$ by $a_h(y-\tilde{y}_h,\chi)=0$ for $\chi \in V_h$. Then, by the triangle inequality, we have

$$\|y-y_h\|^2_{L^2(\Omega)} \le \underbrace{\|y-\tilde{y}_h\|^2_{L^2(\Omega)}}_{K_1} + \underbrace{\|\tilde{y}_h-y_h\|^2_{L^2(\Omega)}}_{K_2}.$$

K_1

Consider the following equation,

$$\begin{aligned}
-\Delta t - \nabla\cdot(\vec{\beta}t) + ct &= y-\tilde{y}_h \text{ in } \Omega \\
t &= 0 \qquad \text{on } \Gamma.
\end{aligned}$$

By the boundedness of the bilinear form and using the Galerkin orthogonality,

$$\begin{aligned}
\|y-\tilde{y}_h\|^2_{L^2(\Omega)} &= a_h(y-\tilde{y}_h, t) \\
&= a_h(y-\tilde{y}_h, t-P_h t) + \underbrace{a_h(y-\tilde{y}_h, P_h t)}_{=0} \\
&\le C|||t-P_h t|||.|||y-\tilde{y}_h||| \le Ch\|t\|_{H^2(\Omega)}|||y-\tilde{y}_h|||.
\end{aligned}$$

By using Theorem (2.2) and Lemma (5.3), we obtain

$$K_1 \leq Ch\|y - \tilde{y}_h\|_{L^2(\Omega)}\|y\|_{H^1(\Omega)} \leq \frac{1}{4}\|y - \tilde{y}_h\|^2_{L^2(\Omega)} + Ch^2\|y\|^2_{H^1(\Omega)}.$$

By canceling $\|y - \tilde{y}_h\|^2_{L^2(\Omega)}$, we obtain that

$$K_1 \leq Ch^2\|y\|^2_{H^1(\Omega)}.$$

K_2 :
Let us define another dual equation,

$$\begin{aligned} -\Delta v - \nabla \cdot (\vec{\beta} v) + cv &= \tilde{y}_h - y_h \quad \text{in } \Omega \\ v &= 0 \qquad\qquad \text{on } \Gamma. \end{aligned}$$

$$\begin{aligned} \|\tilde{y}_h - y_h\|^2_{L^2(\Omega)} &= a_h(\tilde{y}_h - y_h, v) \\ &= \underbrace{a_h(\tilde{y}_h - y, v)}_{K_{21}} + \underbrace{a_h(y - y_h, v)}_{K_{22}}. \end{aligned}$$

K_{21} :
Likewise K_1,

$$\begin{aligned} K_{21} &= a_h(\tilde{y}_h - y, v) = a_h(\tilde{y}_h - y, v - P_h v) + \underbrace{a_h(\tilde{y}_h - y, P_h v)}_{=0} \\ &\leq C|||v - P_h v||| |||\tilde{y}_h - y||| \leq Ch\|v\|_{H^2(\Omega)}|||\tilde{y}_h - y|||. \end{aligned}$$

By using Theorem (2.2) and Lemma (5.3), we obtain

$$K_{21} \leq Ch\|\tilde{y}_h - y_h\|_{L^2(\Omega)}\|y\|_{H^1(\Omega)}.$$

K_{22} :
$K_{22} = a_h(y - y_h, v) = \underbrace{a_h(y - y_h, v - P_h v)}_{K_{221}} + \underbrace{a_h(y - y_h, P_h v)}_{K_{222}}.$
By using the boundedness of the bilinear form, Theorem (2.2) and Lemma (5.3),

$$\begin{aligned} K_{221} &= a_h(y - y_h, v - P_h v) \leq C|||v - P_h v||| |||y - y_h||| \leq Ch\|v\|_{H^2(\Omega)}|||y - y_h||| \\ &\leq Ch\|\tilde{y}_h - y_h\|_{L^2(\Omega)}|||y - y_h|||. \end{aligned}$$

By using Lemma (5.6), we obtain

$$\begin{aligned} K_{221} &\leq Ch\|\tilde{y}_h - y_h\|_{L^2(\Omega)}|||y - y_h||| \\ &\leq Ch\|\tilde{y}_h - y_h\|_{L^2(\Omega)}(h^{-1/2}\|q - q_h\|_{L^2(\Gamma)} + \|y\|_{H^1(\Omega)}) \\ &\leq C\|\tilde{y}_h - y_h\|_{L^2(\Omega)}(h^{1/2}\|q - q_h\|_{L^2(\Gamma)} + h\|y\|_{H^1(\Omega)}). \end{aligned}$$

K_{222} :
Using the fact that $v = 0$ on Γ, Theorems (2.2), (3.2) and Lemma (5.3), we have that

$$\begin{aligned} K_{222} &= a_h(y - y_h, P_h v) = \ell_h(0; q - q_h, P_h v) \leq Ch^{-1}\|q - q_h\|_{L^2(\Gamma)}\|P_h v\|_{L^2(\Gamma)} \\ &\leq Ch^{-1}\|q - q_h\|_{L^2(\Gamma)}\|[[P_h v - v]]\|_{L^2(\Gamma)} \leq Ch^{-1}\|q - q_h\|_{L^2(\Gamma)}h^{1/2}|||P_h v - v||| \\ &\leq Ch^{-1}\|q - q_h\|_{L^2(\Gamma)}h^{1/2}h\|v\|_{H^2(\Omega)} \leq Ch^{-1}\|q - q_h\|_{L^2(\Gamma)}h^{3/2}\|\tilde{y}_h - y_h\|_{L^2(\Omega)}. \end{aligned}$$

Then, we obtain

$$K_{222} \leq Ch^{1/2}\|q - q_h\|_{L^2(\Gamma)}\|\tilde{y}_h - y_h\|_{L^2(\Omega)}.$$

Thus, we have

$$\begin{aligned}\|\tilde{y}_h - y_h\|^2_{L^2(\Omega)} &\leq K_{21} + \underbrace{K_{22}}_{K_{221}+K_{222}} \\ &\leq Ch^2\|y\|_{H^1(\Omega)} + C\|\tilde{y}_h - y_h\|_{L^2(\Omega)}(h^{1/2}\|q - q_h\|_{L^2(\Gamma)} + h\|y\|_{H^1(\Omega)}) \\ &\leq \frac{1}{4}\|\tilde{y}_h - y_h\|^2_{L^2(\Omega)} + Ch^2\|y\|^2_{H^1(\Omega)} + Ch\|q - q_h\|^2_{L^2(\Gamma)}.\end{aligned}$$

By canceling $\|\tilde{y}_h - y_h\|^2_{L^2(\Omega)}$, we obtain

$$\|\tilde{y}_h - y_h\|^2_{L^2(\Omega)} \leq Ch^2\|y\|^2_{H^1(\Omega)} + Ch\|q - q_h\|^2_{L^2(\Gamma)}.$$

Finally, we obtain

$$\begin{aligned}\|y - y_h\|^2_{L^2(\Omega)} &\leq \|y - \tilde{y}_h\|^2_{L^2(\Omega)} + \|\tilde{y}_h - y_h\|^2_{L^2(\Omega)} \\ &\leq C(h\|q - q_h\|^2_{L^2(\Gamma)} + h^2\|y\|^2_{H^1(\Omega)}).\end{aligned}$$

By taking the square root, we conclude

$$\|y - y_h\|_{L^2(\Omega)} \leq C(h^{1/2}\|q - q_h\|_{L^2(\Gamma)} + h\|y\|_{H^1(\Omega)}).$$

□

5.2. *Main results*

Now, we are ready to prove the main result of the paper. We will state it in the next theorem.

Theorem 5.1. *Let Ω be a convex polygonal domain, $\bar{q}$ be the optimal control of the problem* (1.1) *and $\bar{q}_h$ be its optimal SIPG solution. Then, for h sufficiently small, there exists a positive constant C independent of h such that*

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} \leq Ch^{1/2}(|\bar{q}|_{H^{1/2}(\Gamma)} + \|\bar{y}\|_{H^1(\Omega)} + \|\hat{y}\|_{L^2(\Omega)}), \tag{5.3}$$

where $(\bar{y}, \bar{q}, \bar{z}) \in H^1(\Omega) \times H^{1/2}(\Gamma) \times H^2(\Omega)$ and $\hat{y} \in L^2(\Omega)$.

Proof. Since $\bar{q}$ is the optimal solution of the problem (1.1) and $\bar{q}$ satisfies the Eq (3.3), we have

$$\alpha\langle\bar{q}, \phi_h\rangle_\Gamma + \langle\phi_h, \frac{\partial\bar{z}}{\partial n}\rangle_\Gamma = 0, \quad \forall\phi_h \in Q_h. \tag{5.4}$$

Since $\bar{q}_h$ is the approximate solution of the problem (1.1) and $\bar{q}_h$ satisfies the Eq (4.8), we have

$$\alpha\langle\bar{q}_h, \phi_h\rangle_\Gamma + \langle\phi_h, \frac{\partial\bar{z}_h}{\partial n}\rangle_\Gamma - \frac{\gamma}{h}\langle\phi_h, \bar{z}_h\rangle_\Gamma - \langle\bar{z}_h|\vec{n}\cdot\vec{\beta}|, \phi_h\rangle_{\Gamma^-} = 0, \quad \forall\phi_h \in Q_h. \tag{5.5}$$

Subtracting the Eq (5.4) from the Eq (5.5), for any $\phi_h \in Q_h$, we have

$$\alpha\langle\bar{q} - \bar{q}_h, \phi_h\rangle_\Gamma + \langle\phi_h, \frac{\partial(\bar{z} - \bar{z}_h)}{\partial n}\rangle_\Gamma + \frac{\gamma}{h}\langle\phi_h, \bar{z}_h\rangle_\Gamma + \langle\bar{z}_h|\vec{n}\cdot\vec{\beta}|, \phi_h\rangle_{\Gamma^-} = 0. \tag{5.6}$$

Taking $\phi_h = P_h^{\partial}(\bar{q} - \bar{q}_h) = P_h^{\partial}\bar{q} - P_h^{\partial}\bar{q}_h = P_h^{\partial}\bar{q} - \bar{q}_h$ in the Eq (5.6) and splitting

$$P_h^{\partial}\bar{q} - \bar{q}_h = (P_h^{\partial}\bar{q} - \bar{q}) + (\bar{q} - \bar{q}_h),$$

we obtain

$$\begin{aligned}
\alpha\|\bar{q} - \bar{q}_h\|^2_{L^2(\Gamma)} &= \alpha\langle \bar{q} - \bar{q}_h, \bar{q} - \bar{q}_h\rangle \\
&\leq \underbrace{\boxed{\alpha\langle \bar{q} - \bar{q}_h, P_h^{\partial}\bar{q} - \bar{q}\rangle_\Gamma}}_{J_1} + \underbrace{\boxed{\langle P_h^{\partial}\bar{q} - \bar{q}, \frac{\partial(\bar{z} - \bar{z}_h)}{\partial n}\rangle_\Gamma}}_{J_2} \\
&+ \underbrace{\boxed{\frac{\gamma}{h}\langle P_h^{\partial}\bar{q} - \bar{q}, \bar{z}_h\rangle_\Gamma}}_{J_3} + \underbrace{\boxed{\langle P_h^{\partial}\bar{q} - \bar{q}, \bar{z}_h|\vec{n}\cdot\vec{\beta}|\rangle_{\Gamma^-}}}_{J_4} \\
&+ \underbrace{\boxed{\langle \bar{q} - \bar{q}_h, \frac{\partial(\bar{z} - \bar{z}_h)}{\partial n}\rangle_\Gamma}}_{J_5} + \underbrace{\boxed{\frac{\gamma}{h}\langle \bar{q} - \bar{q}_h, \bar{z}_h\rangle_\Gamma}}_{J_6} \\
&+ \underbrace{\boxed{\langle \bar{q} - \bar{q}_h, \bar{z}_h|\vec{n}\cdot\vec{\beta}|\rangle_{\Gamma^-}}}_{J_7} \\
&= J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7.
\end{aligned} \tag{5.7}$$

Now, we shall estimate each term separately. Most terms can be estimated by using the estimate of the L^2-projection. However, the term $(\bar{z} - \bar{z}_h)$ in J_2 and J_5 is not in the discrete space, so additional arguments are needed to treat these terms.

Estimate for J_1: By the Cauchy-Schwarz inequality and using Lemma (5.4),

$$\begin{aligned}
J_1 = \alpha\langle \bar{q} - \bar{q}_h, P_h^{\partial}\bar{q} - \bar{q}\rangle_\Gamma &\leq \alpha\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)}\|P_h^{\partial}\bar{q} - \bar{q}\|_{L^2(\Gamma)} \\
&\leq C_1 h^{1/2}|\bar{q}|_{H^{1/2}(\Gamma)}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)},
\end{aligned}$$

where C_1 depends on α.

Estimates for J_3 **and** J_6: Using Lemma (5.5) to estimate $\|\bar{z}_h\|_{L^2(\Gamma)}$, the Cauchy-Schwarz inequality, Lemma (5.7) and the regularity of $\bar{y}$, then we have

$$\begin{aligned}
J_3 &= \frac{\gamma}{h}\langle P_h^{\partial}\bar{q} - \bar{q}, \bar{z}_h\rangle_\Gamma \leq \frac{\gamma}{h}\|P_h^{\partial}\bar{q} - \bar{q}\|_{L^2(\Gamma)}\|\bar{z}_h\|_{L^2(\Gamma)}. \\
&\leq C_3 h^{-1}h^{1/2}|\bar{q}|_{H^{1/2}(\Gamma)}h^{3/2}\|\hat{y} - \bar{y}_h\|_{L^2(\Omega)} \\
&\leq C_3 h|\bar{q}|_{H^{1/2}}\|\hat{y} - \bar{y}_h\|_{L^2(\Omega)} \leq C_3 h|q|_{H^{1/2}}(\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}) \\
&\leq C_3 h|q|_{H^{1/2}}(\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + h^{1/2}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} + h\|\bar{y}\|_{H^1(\Omega)}).
\end{aligned}$$

Likewise,

$$\begin{aligned}
J_6 &= \frac{\gamma}{h}\langle \bar{q} - \bar{q}_h, \bar{z}_h\rangle_\Gamma \leq \frac{\gamma}{h}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)}\|\bar{z}_h\|_{L^2(\Gamma)} \\
&\leq C_6 h^{1/2}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)}\|\hat{y} - \bar{y}_h\|_{L^2(\Omega)} \\
&\leq C_6 h^{1/2}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)}(\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}) \\
&\leq C_6 h^{1/2}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)}(\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + h^{1/2}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} + h\|\bar{y}\|_{H^1(\Omega)}),
\end{aligned}$$

where C_3 and C_6 depend on γ.

Estimates for J_4 and J_7: By using the Cauchy-Schwarz inequality, Lemmas (5.5) and (5.7), we have

$$\begin{aligned} J_4 &= \langle P_h^{\partial}\bar{q} - \bar{q}, \bar{z}_h |\vec{n}\cdot\vec{\beta}|\rangle_{\Gamma^-} \leq C_4 h^{1/2}|\bar{q}|_{H^{1/2}(\Gamma)}\|\beta\|^{1/2}_{L^\infty(\Gamma)}\|\bar{z}_h\|_{L^2(\Gamma)} \\ &\leq C_4 h^2 |q|_{H^{1/2}(\Gamma)}\|\beta\|^{1/2}_{L^\infty(\Gamma)}\|\hat{y} - \bar{y}_h\|_{L^2(\Omega)} \\ &\leq C_4 h^2 |\bar{q}|_{H^{1/2}(\Gamma)}\|\beta\|^{1/2}_{L^\infty(\Gamma)}\big(\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}\big) \\ &\leq C_4 h^2 |\bar{q}|_{H^{1/2}(\Gamma)}\|\beta\|^{1/2}_{L^\infty(\Gamma)}\big(\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + h^{1/2}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} + h\|\bar{y}\|_{H^1(\Omega)}\big). \end{aligned}$$

Likewise,

$$\begin{aligned} J_7 &= \langle \bar{q} - \bar{q}_h, \bar{z}_h |\vec{n}\cdot\vec{\beta}|\rangle_{\Gamma^-} \leq C_7\|\beta\|^{1/2}_{L^\infty(\Gamma)}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)}\|\bar{z}_h\|_{L^2(\Gamma)} \\ &\leq C_7\|\beta\|^{1/2}_{L^\infty(\Gamma)}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} h^{3/2}\|\hat{y} - \bar{y}_h\|_{L^2(\Omega)} \\ &\leq C_7 h^{3/2}\|\beta\|^{1/2}_{L^\infty(\Gamma)}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)}\big(\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}\big) \\ &\leq C_7 h^{3/2}\|\beta\|^{1/2}_{L^\infty(\Gamma)}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)}\big(\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + h^{1/2}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} + h\|\bar{y}\|_{H^1(\Omega)}\big). \end{aligned}$$

Estimate for J_5: By the Cauchy-Schwarz inequality,

$$J_5 = \langle \bar{q} - \bar{q}_h, \frac{\partial(\bar{z} - \bar{z}_h)}{\partial n}\rangle_\Gamma \leq \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)}\|\frac{\partial(\bar{z} - \bar{z}_h)}{\partial n}\|_{L^2(\Gamma)}.$$

Let us define $\tilde{z}_h \in V_h$ to be the SIPG solution to $\bar{z}$ i.e. $a_h(\chi, \tilde{z}_h) = (\hat{y} - \bar{y}, \chi)$, $\forall \chi \in V_h$.

In particular, $a_h(\chi, \bar{z} - \tilde{z}_h) = 0$ by the Galerkin orthogonality. Thus, we continue as following,

$$\langle \bar{q} - \bar{q}_h, \frac{\partial(\bar{z} - \bar{z}_h)}{\partial n}\rangle_\Gamma \leq \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)}\left(\underbrace{\|\frac{\partial(\bar{z} - \tilde{z}_h)}{\partial n}\|_{L^2(\Gamma)}}_{J_{51}} + \underbrace{\|\frac{\partial(\tilde{z}_h - \bar{z}_h)}{\partial n}\|_{L^2(\Gamma)}}_{J_{52}}\right).$$

J_{51}:

By the triangle inequality, we have

$$\|\frac{\partial(\bar{z} - \tilde{z}_h)}{\partial n}\|_{L^2(\Gamma)} \leq \underbrace{\|\frac{\partial(\bar{z} - P_h\bar{z})}{\partial n}\|_{L^2(\Gamma)}}_{J_{511}} + \underbrace{\|\frac{\partial(P_h\bar{z} - \tilde{z}_h)}{\partial n}\|_{L^2(\Gamma)}}_{J_{512}}.$$

J_{511} :

By the trace inequality, Theorem (2.2) and Lemma (5.2), we obtain

$$\begin{aligned} J_{511} &= \|\frac{\partial(\bar{z} - P_h\bar{z})}{\partial n}\|^2_{L^2(\Gamma)} = \sum_{e\in\Gamma}\|\frac{\partial(\bar{z} - P_h\bar{z})}{\partial n}\|^2_{L^2(e)} \\ &\leq \sum_{\tau\in T_h}(Ch^{-1}\|\bar{z} - P_h\bar{z}\|^2_{H^1(\tau)} + Ch\|\bar{z} - P_h\bar{z}\|^2_{H^2(\tau)}) \\ &\leq \sum_{\tau\in T_h} Ch\|\bar{z}\|^2_{H^2(\tau)} = Ch\|\bar{z}\|^2_{H^2(\Omega)} \leq Ch\|\hat{y} - \bar{y}\|^2_{L^2(\Omega)}. \end{aligned}$$

Thus,

$$J_{511} = \|\frac{\partial(\bar{z} - P_h\bar{z})}{\partial n}\|_{L^2(\Gamma)} \leq Ch^{1/2}\|\hat{y} - \bar{y}\|_{L^2(\Omega)}.$$

J_{512} :
Since $(P_h\bar{z} - \tilde{z}_h) \in V_h$, we can apply the trace theorem for discrete function and by using the inverse inequality and Lemma (5.2), we obtain that

$$\begin{aligned}\|\frac{\partial(P_h\bar{z} - \tilde{z}_h)}{\partial n}\|_{L^2(\Gamma)} &\le Ch^{-1/2}\|P_h\bar{z} - \tilde{z}_h\|_{H^1(\Omega)} \\ &\le Ch^{-1/2}\|\!|P_h\bar{z} - \tilde{z}_h\|\!| \le Ch^{-1/2}(\|\!|P_h\bar{z} - \bar{z}\|\!| + \|\!|\bar{z} - \tilde{z}_h\|\!|) \\ &\le Ch^{-1/2}h\|\bar{z}\|_{H^2(\Omega)} \le Ch^{1/2}\|\hat{y} - \bar{y}\|_{L^2(\Omega)},\end{aligned}$$

where we have used Lemma (5.3) for $k = 1$ by Theorem (2.2) and Lemma (5.2).
Thus,

$$\|\frac{\partial(P_h\bar{z} - \tilde{z}_h)}{\partial n}\|_{L^2(\Gamma)} \le Ch^{1/2}\|\hat{y} - \bar{y}\|_{L^2(\Omega)}.$$

Since $J_{51} = J_{511} + J_{512}$, we obtain

$$J_{51} = \|\frac{\partial(\bar{z} - \tilde{z}_h)}{\partial n}\|_{L^2(\Gamma)} \le Ch^{1/2}\|\hat{y} - \bar{y}\|_{L^2(\Omega)} \le Ch^{1/2}(\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)}).$$

J_{52} :
Since we have

$$\begin{aligned}a_h(\chi, \bar{z}_h) &= (\hat{y} - \bar{y}_h, \chi), \\ a_h(\chi, \tilde{z}_h) &= (\hat{y} - \bar{y}, \chi),\end{aligned}$$

where $\forall\chi \in V_h$. We obtain

$$a_h(\chi, \tilde{z}_h - \bar{z}_h) = (\bar{y}_h - \bar{y}, \chi), \qquad \forall\chi \in V_h. \tag{5.8}$$

Now, let us define the following equation

$$\begin{aligned}-\Delta w - \nabla \cdot (\vec{\beta} w) + cw &= \bar{y}_h - \bar{y} \quad \text{in } \Omega \\ w &= 0 \qquad \text{on } \Gamma.\end{aligned}$$

By using the Eq (5.8),

$$a_h(\chi, \tilde{z}_h - \bar{z}_h) = a_h(\chi, \tilde{z}_h) - a_h(\chi, \bar{z}_h) = (\hat{y} - \bar{y}, \chi) - (\hat{y} - \bar{y}_h, \chi) = (\bar{y}_h - \bar{y}, \chi) = a_h(\chi, w_h).$$

The above equality shows that $w_h = \tilde{z}_h - \bar{z}_h$.

Now, using the inverse inequality and the fact that $w = 0$ on Γ, we obtain

$$\begin{aligned}\|\frac{\partial(\tilde{z}_h - \bar{z}_h)}{\partial n}\|_{L^2(\Gamma)} &\le Ch^{-1}\|\tilde{z}_h - \bar{z}_h\|_{L^2(\Gamma)} = Ch^{-1}\|\tilde{z}_h - \bar{z}_h - w\|_{L^2(\Gamma)} \\ &\le Ch^{-1}h^{1/2}\|\!|(\tilde{z}_h - \bar{z}_h) - w\|\!| \\ &\le Ch^{-1/2}\|\!|w_h - w\|\!| \le Ch^{-1/2}h\|w\|_{H^2(\Omega)} \\ &\le Ch^{1/2}\|\bar{y}_h - \bar{y}\|_{L^2(\Omega)} \\ &\le Ch^{1/2}(h^{1/2}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} + h\|\bar{y}\|_{H^1(\Omega)}),\end{aligned}$$

where we have used Theorem (2.2) and Lemma (5.3) for $k = 1$ by Lemmas (5.2) and (5.7) in the last step.
Thus,

$$J_{52} = \|\frac{\partial(\tilde{z}_h - \bar{z}_h)}{\partial n}\|_{L^2(\Gamma)} \leq Ch^{1/2}(h^{1/2}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} + h\|\bar{y}\|_{H^1(\Omega)}).$$

Finally, we obtain

$$J_5 \leq \|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)}(J_{51} + J_{52}) \leq C_5 h^{1/2}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)}(\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + h^{1/2}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} + h\|\bar{y}\|_{H^1(\Omega)}).$$

Estimate for J_2: By using the the estimation of $\|\frac{\partial(\bar{z}-\bar{z}_h)}{\partial n}\|_{L^2(\Gamma)}$ in J_5, Cauchy-Schwarz inequality and Lemma (5.4), we have

$$\begin{aligned} J_2 = \langle P_h^{\partial}\bar{q} - \bar{q}, \frac{\partial(\bar{z} - \bar{z}_h)}{\partial n}\rangle_\Gamma &\leq \|P_h^{\partial}\bar{q} - \bar{q}\|_{L^2(\Gamma)}\|\frac{\partial(\bar{z} - \bar{z}_h)}{\partial n}\|_{L^2(\Gamma)} \\ &\leq C_2 h^{1/2}|\bar{q}|_{H^{1/2}(\Gamma)}\|\frac{\partial(\bar{z} - \bar{z}_h)}{\partial n}\|_{L^2(\Gamma)} \\ &\leq C_2 h^{1/2}|\bar{q}|_{H^{1/2}(\Gamma)} h^{1/2}(\|\hat{y}\|_{L(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + h^{1/2}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} + h\|\bar{y}\|_{H^1(\Omega)}) \\ &= C_2 h|\bar{q}|_{H^{1/2}(\Gamma)}(\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + h^{1/2}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} + h\|\bar{y}\|_{H^1(\Omega)}). \end{aligned}$$

Thus,

$$J_2 \leq C_2 h|q|_{H^{1/2}(\Gamma)}(\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + h^{1/2}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} + h\|\bar{y}\|_{H^1(\Omega)}).$$

After using Lemma (5.7) to estimate $\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}$ and combining $J_1, J_2, J_3, J_4, J_5, J_6, J_7$ in the Eq (5.7), we obtain

$$\begin{aligned} &\alpha\|\bar{q} - \bar{q}_h\|^2_{L^2(\Omega)} \leq J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 \\ &\leq C_1 h^{\frac{1}{2}}|\bar{q}|_{H^{1/2}(\Gamma)}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} \\ &+ C_2 h|\bar{q}|_{H^{1/2}(\Gamma)}(\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + h^{1/2}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} + h\|\bar{y}\|_{H^1(\Omega)}) \\ &+ C_3 h|\bar{q}|_{H^{1/2}}(\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + h^{1/2}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} + h\|\bar{y}\|_{H^1(\Omega)}) \\ &+ C_4 h^2|\bar{q}|_{H^{1/2}(\Gamma)}\|\beta\|^{\frac{1}{2}}_{L^\infty(\Gamma)}(\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + h^{1/2}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} + h\|\bar{y}\|_{H^1(\Omega)}) \\ &+ C_5 h^{\frac{1}{2}}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)}(\|\hat{y}\|_{H^1(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + h^{1/2}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} + h\|\bar{y}\|_{H^1(\Omega)}) \\ &+ C_6 h^{\frac{1}{2}}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)}(\|\hat{y}\|_{H^1(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + h^{1/2}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} + h\|\bar{y}\|_{H^1(\Omega)}) \\ &+ C_7 h^{\frac{3}{2}}\|\beta\|^{\frac{1}{2}}_{L^\infty(\Gamma)}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)}\big(\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + h^{1/2}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} + h\|\bar{y}\|_{H^1(\Omega)}\big). \end{aligned}$$

Notice that we can rewrite the above inequality as

$$\begin{aligned} &\alpha\|\bar{q} - \bar{q}_h\|^2_{L^2(\Gamma)} \leq C_I h|\bar{q}|^2_{H^{1/2}(\Gamma)} + \frac{\alpha}{4}\|\bar{q} - \bar{q}_h\|^2_{L^2(\Gamma)} \\ &+ C_{II} h(\|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)} + h^{1/2}\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} + h\|\bar{y}\|_{H^1(\Omega)})^2 + \frac{\alpha}{4}\|\bar{q} - \bar{q}_h\|^2_{L^2(\Gamma)} \\ &+ C_{III} h\|\bar{q} - \bar{q}_h\|^2_{L^2(\Gamma)}. \end{aligned}$$

After all simplification, we obtain

$$\alpha\|\bar{q} - \bar{q}_h\|^2_{L^2(\Gamma)} \leq Ch(|\bar{q}|_{H^{1/2}(\Gamma)} + \|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)})^2 + C'h\|\bar{q} - \bar{q}_h\|^2_{L^2(\Gamma)},$$

where h is sufficiently small such that $C'h \leq \frac{\alpha}{2}$ to absorb $C'h\|\bar{q} - \bar{q}_h\|^2_{L^2(\Gamma)}$ to the left hand side. Thus, we conclude that there exists a positive constant C such that

$$\|\bar{q}-\bar{q}_h\|_{L^2(\Gamma)} \leq Ch^{1/2}(|\bar{q}|_{H^{1/2}(\Gamma)} + \|\hat{y}\|_{L^2(\Omega)} + \|\bar{y}\|_{H^1(\Omega)}),$$

provided h is sufficiently small. □

6. Numerical examples

In this section, we show the features of the method and some numerical examples to support our theoretical results by the method described for the main problem (1.1), (1.2a) and (1.2b). Here, we present numerical results depending on different kinds of domain as the following.

6.1. Ω is a line segmenent

Since the domain is one dimensional and the boundary is consisting of two points, there is no regularity limitation due to geometry restriction. Thus, we do not expect an optimal convergence rate, but we observe that the method is still stable and convergent in Tables 1–3 and Figures 3–4.

By setting $\Omega = [0,1]$, $\epsilon = 1$, $\alpha = 1$, $\vec{\beta} = [1]$, $\bar{q} = (1-x)^2(x^2)$, $c = 0$, $\bar{y} = x^4 - \frac{e^{\frac{x-1}{\epsilon}} - e^{\frac{-1}{\epsilon}}}{1-e^{\frac{-1}{\epsilon}}}$, and $\bar{z} = \frac{\alpha}{\epsilon}(1-x)^2x^2$.

Table 1. 1D Error rates for piecewise linear basis functions.

h	$L^2 - y_{rate}$	$H^1 - y_{rate}$	$Left - q_{rate}$	$Right - q_{rate}$
5.00e-01	1.959	1.002	2.007	1.965
2.50e-01	1.979	1.001	2.006	1.982
1.25e-01	1.990	1.001	2.004	1.991
6.25e-02	1.995	1.000	2.002	1.996
3.12e-02	1.997	1.001	2.001	1.998
1.56e-03	1.999	1.000	2.001	1.999

Table 2. 1D Error rates for piecewise quadratic basis functions.

h	$L^2 - y_{rate}$	$H^1 - y_{rate}$	$Left - q_{rate}$	$Right - q_{rate}$
5.00e-01	2.999	2.013	2.025	2.982
2.50e-01	2.999	2.007	2.683	2.991
1.25e-01	3.000	2.004	2.864	2.996
6.25e-02	3.000	2.002	2.937	2.998
3.12e-02	2.998	2.001	2.969	2.999
1.56e-03	1.938	2.001	2.985	2.999

Table 3. 1D Error rates for piecewise cubic basis functions.

h	$L^2 - y_{rate}$	$H^1 - y_{rate}$	$Left - q_{rate}$	$Right - q_{rate}$
5.00e-01	4.436	2.436	3.970	3.866
2.50e-01	4.425	3.423	3.985	3.983
1.25e-01	4.380	3.385	3.992	3.991
6.25e-02	1.188	3.320	3.996	3.996
3.12e-02	-0.777	3.223	3.998	3.998
1.56e-03	-1.119	1.268	3.999	3.999

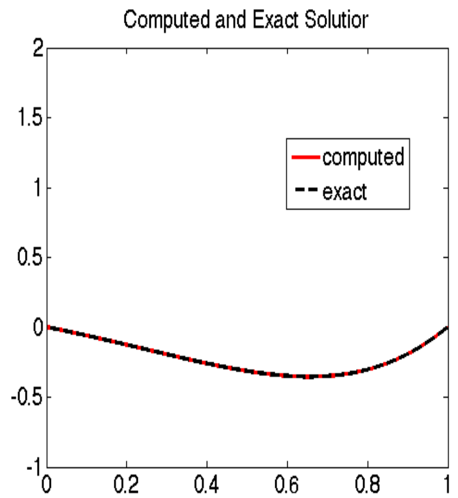

Figure 3. Computed and exact state solution.

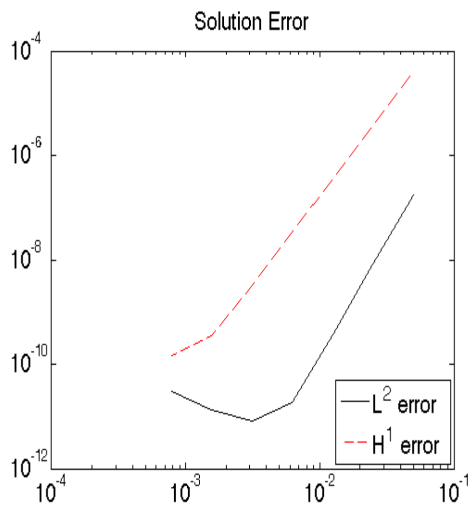

Figure 4. State solution error.

6.2. Ω is a unit square domain

By setting the problem as the following,

$$\Omega = [0,1]\times[0,1],\ \vec{\beta} = [1;1],\ c = 1, \alpha = 1,\ \bar{q} = \frac{-1}{\epsilon}(x(1-x)+y(1-y)),$$
$$\bar{y} = \frac{-1}{\epsilon}(x(1-x)+y(1-y)),\ \bar{z} = \frac{\alpha}{\epsilon}(xy(1-x)(1-y)).$$

Here, we consider piecewise linear continuous functions to approximate the optimal control.

6.2.1. Numerical results for the regular case ($\epsilon \gg h$ where h is mesh size) on the unit square domain

The first order conditions allow us deduce the regularity results of the optimal control and so the expected convergence rate has agreed well with the rate in [6] as $\|\bar{q}-\bar{q}_h\|_{L^2(\Gamma)} \le Ch$ by the square domain with the largest interior angle $w_{max} = \frac{\pi}{2}$.

Lemma (5.7) and $\|\bar{q}-\bar{q}_h\|_{L^2(\Gamma)} \le Ch$ yield $\|\bar{y}-\bar{y}_h\|_{L^2(\Omega)} \le Ch^{3/2}$. Since the power of h on the right-hand side drops for one for each derivative of the error $(\bar{y}-\bar{y}_h)$, $\|\bar{y}-\bar{y}_h\|_{H^1(\Omega)} \le Ch^{1/2}$ by Lemma (5.2). Likewise, From Lemma (5.2) and $z \in H^2$, $\|\bar{z}-\bar{z}_h\|_{L^2(\Omega)} \le Ch^2$, and so $\|\bar{z}-\bar{z}_h\|_{H^1(\Omega)} \le Ch$ as our expected convergence rates indicated in Tables 4 and 5. Also, the expected rates have agreed well with the rates for the different densities of the meshes in [6]. Also, we can see the stability of the method in Figure 5.

Table 4. Error for the regular case on the unit square domain.

h	$\|\bar{y}-\bar{y}_h\|_{L^2}$	$\|\bar{y}-\bar{y}_h\|_{H^1}$	$\|\bar{q}-\bar{q}_h\|_{L^2}$	$\|\bar{z}-\bar{z}_h\|_{L^2}$	$\|\bar{z}-\bar{z}_h\|_{H^1}$
5.00e-01	1.92e-01	3.91e+00	1.07e+00	4.31e-02	9.19e-01
2.50e-01	8.44e-02	2.25e+00	4.86e-01	1.38e-02	5.06e-01
1.25e-01	3.14e-02	1.27e+00	2.29e-01	4.21e-03	2.62e-01
6.25e-02	1.10e-02	7.33e-01	1.09e-01	1.22e-03	1.32e-01
3.12e-02	3.80e-03	4.49e-01	5.20e-02	3.32e-04	6.66e-02

Table 5. Error rates for the regular case on the unit square domain.

h	L^2-y_{rate}	H^1-y_{rate}	L^2-q_{rate}	L^2-z_{rate}	H^1-z_{rate}
5.00e-01	0.00	0.00	0.00	0.00	0.00
2.50e-01	1.19	0.80	1.13	1.64	0.86
1.25e-01	1.43	0.83	1.08	1.72	0.95
6.25e-02	1.51	0.79	1.08	1.79	0.98
3.12e-02	1.53	0.71	1.06	1.87	0.99
expected	1.50	0.50	1.00	2.00	1.00

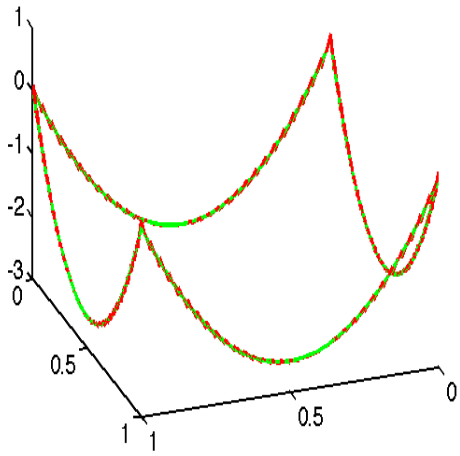

Figure 5. Exact and computed control for the regular case on the unit square domain are shown in green and red, respectively.

6.2.2. Numerical results for the advection-diffusion dominated case ($h \gg \epsilon$ where h is mesh size) on the unit square domain

Since ϵ is too small for this case such as $\epsilon = 10^{-5}$, $\frac{h|\vec{\beta}|}{\epsilon} > 1$ which means the advection-diffusion dominated case occurs. The norm of y depends on ϵ such that $||\bar{y}||_{H^{k+1}(\Omega)} \leq \frac{C}{\epsilon^{k+1/2}}$. Since the convergence rate of $\bar{q}$ depends on data of $\bar{y}$ from the main result, we do not expect any convergence rate and so this case does not contradict with our main result. Also, the feature of the method shows itself that Dirichlet boundary condition is almost ignored by the method as a result of weak treatment and it does not resolve the layers and causes oscillations on the boundary. It can be seen in Figure 6 and Tables 6 and 7 that some oscillatory solutions and non-convergent rate of q appear on the inflow boundary, caused by non-stabilized terms of boundary edges E_h^{∂} represented by E_h^{-} in the bilinear form (4.6) and (4.5), whereas it is stable on both the interior edges E_h^0 and the stabilized boundary edges E_h^{∂} by the penalty term in the form.

Table 6. Error for the the advection-diffusion dominated case on the unit square domain.

h	$\|\|\bar{y} - \bar{y}_h\|\|_{L^2}$	$\|\|\bar{y} - \bar{y}_h\|\|_{H^1}$	$\|\|\bar{q} - \bar{q}_h\|\|_{L^2}$	$\|\|\bar{z} - \bar{z}_h\|\|_{L^2}$	$\|\|\bar{z} - \bar{z}_h\|\|_{H^1}$
5.00e-01	4.29e+00	2.68e+01	8.15e+00	5.66e+01	9.52e+02
2.50e-01	7.69e-01	1.32e+01	2.22e+00	1.51e+01	5.15e+02
1.25e-01	4.72e-01	1.58e+01	9.70e-01	3.85e+00	2.63e+02
6.25e-02	3.78e-01	2.36e+01	6.66e-01	9.64e-01	1.32e+02
3.12e-02	2.55e-01	2.88e+01	6.37e-01	2.40e-01	6.62e+01

Table 7. Error rates for the the advection-diffusion dominated case on the unit square domain.

h	$L^2 - y_{rate}$	$H^1 - y_{rate}$	$L^2 - q_{rate}$	$L^2 - z_{rate}$	$H^1 - z_{rate}$
5.00e-01	0.00	0.00	0.00	0.00	0.00
2.50e-01	2.48	1.02	1.87	1.91	0.89
1.25e-01	0.71	-0.26	1.20	1.97	0.97
6.25e-02	0.32	-0.58	0.54	2.00	0.99
3.12e-02	0.57	-0.28	0.06	2.01	1.00

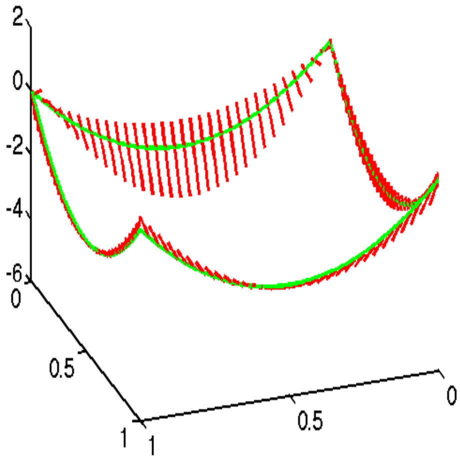

Figure 6. Exact and computed control for the the advection-diffusion dominated case on the unit square domain are shown in green and red, respectively.

6.3. Ω is a diamond shaped domain

By a transformation from the unit square domain to obtain a diamond shaped domain Ω with $\frac{\pi}{4}$,$\frac{\pi}{8}$ and $\frac{\pi}{10}$ angles, while the angle of the domain is getting smaller, we expect that the error rate is getting close to the predicted optimal error rate.

6.3.1. Numerical results for the regular case on the diamond shape domain with angles $\pi/4$, $\pi/8$ and $\pi/10$

After the transformation from the unit square domain to obtain a diamond shaped domain Ω with $\frac{\pi}{4}$,$\frac{\pi}{8}$ and $\frac{\pi}{10}$ angles, we can observe from tables 8–10 that the regularity of the state is reducing sharply close to 1 and that we will obtain the predicted rate i.e., $||\bar{q} - \bar{q}_h||_{L^2(\Gamma)} \leq Ch^{1/2}$ yields $||\bar{y} - \bar{y}_h||_{L^2(\Omega)} \leq Ch^1$ by Theorem (5.7). There are many researches, see [6, 31, 46], which obtained an error estimate for the optimal control of order depends on the largest angle of the boundary polygon. Also, we can see the stable behavior of the method in Figures 7–9.

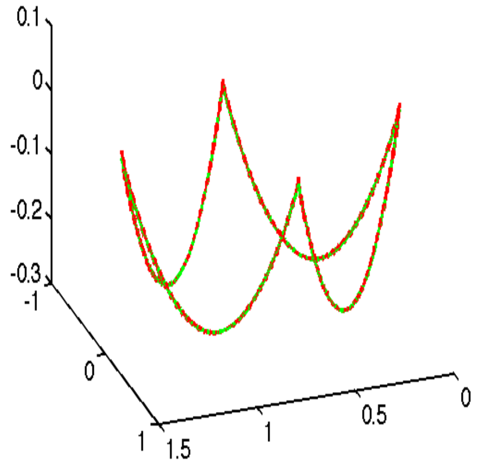

Figure 7. Exact and computed control for the regular case with angle $\pi/4$ are shown in green and red, respectively.

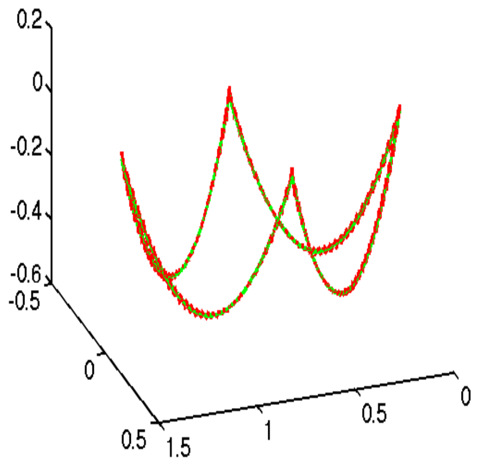

Figure 8. Exact and computed control for the regular case with angle $\pi/8$ are shown in green and red, respectively.

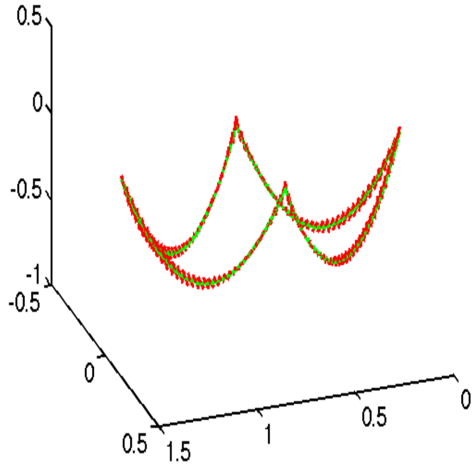

Figure 9. Exact and computed control for the regular case with angle $\pi/10$ are shown in green and red, respectively.

Table 8. Error rates for the regular case with angle on the diamond shape domain with angle $\pi/4$.

h	$L^2 - y_{rate}$	$H^1 - y_{rate}$	$L^2 - q_{rate}$	$L^2 - z_{rate}$	$H^1 - z_{rate}$
5.00e-01	0.00	0.00	0.00	0.00	0.00
2.50e-01	1.51	0.80	0.65	1.43	1.78
1.25e-01	1.47	0.84	0.96	1.75	1.91
6.25e-02	1.69	0.85	1.05	1.89	1.96
3.12e-02	1.75	0.81	1.05	1.95	1.98
1.56e-02	1.73	0.74	1.03	1.97	1.99

Table 9. Error rates for the regular case with angle on the diamond shape domain with angle $\pi/8$.

h	$L^2 - y_{rate}$	$H^1 - y_{rate}$	$L^2 - q_{rate}$	$L^2 - z_{rate}$	$H^1 - z_{rate}$
5.00e-01	0.00	0.00	0.00	0.00	0.00
2.50e-01	1.75	0.72	0.32	1.11	0.76
1.25e-01	1.16	0.41	0.72	1.63	0.95
6.25e-02	1.60	0.50	0.97	1.88	0.99
3.12e-02	1.70	0.55	1.01	1.96	1.00
1.56e-02	1.66	0.53	1.02	1.99	1.00

Table 10. Error rates for the regular case with angle on the diamond shape domain with angle $\pi/10$.

h	$L^2 - y_{rate}$	$H^1 - y_{rate}$	$L^2 - q_{rate}$	$L^2 - z_{rate}$	$H^1 - z_{rate}$
5.00e-01	0.00	0.00	0.00	0.00	0.00
2.50e-01	0.85	0.17	0.17	1.03	0.61
1.25e-01	0.92	0.17	1.59	1.60	1.04
6.25e-02	1.32	0.26	0.90	1.89	1.14
3.12e-02	1.50	0.42	1.00	1.98	1.10
1.56e-02	1.55	0.50	1.02	2.00	1.05

6.3.2. Numerical results for the advection-diffusion dominated case on the diamond shape domain with angle $\pi/4$

While the method still works, likewise the frame in the unit square domain, it can be seen in Figure 10 and Table 11 that some oscillatory solutions and non-convergent rate of q appear on the inflow boundary whereas it is stable on the interior edges and the stabilized boundary edges as a result of weak treatment because of not resolving the layers and causing oscillations on the boundary.

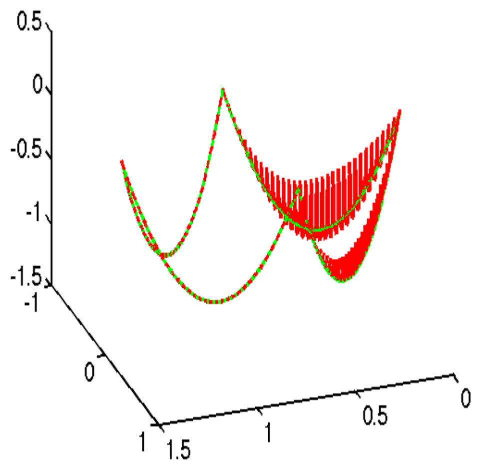

Figure 10. Exact and computed control for the advection-diffusion dominated case on the diamond shape domain with angle $\pi/4$ are shown in green and red, respectively.

Table 11. Error rates for the the advection-diffusion dominated case on the diamond shape domain with angle $\pi/4$.

h	$L^2 - y_{rate}$	$H^1 - y_{rate}$	$L^2 - q_{rate}$	$L^2 - z_{rate}$	$H^1 - z_{rate}$
5.00e-01	0.00	0.00	0.00	0.00	0.00
2.50e-01	3.35	2.45	3.36	2.87	1.89
1.25e-01	0.31	0.23	1.41	2.96	1.97
6.25e-02	-0.01	-0.64	0.81	2.97	1.99
3.12e-02	0.01	-0.68	0.45	2.45	2.00

7. Conclusions

In this paper, we consider Dirichlet boundary optimal control problem governed by the advection-diffusion equation and apply the DG methods to the problem. We show some attractive features of the method such as the stable behavior of the SIPG method into the domain of the smoothness and for the advection dominated case except on the boundary as a result of the boundary weak treatments. We have proven that the convergence rate for the SIPG method is optimal in the interior of the general convex domain. However, all convergence rates in numerical examples are higher than predicted by the main result because the predicted order exists for general convex domain, but obtaining the predicted optimal convergence rate depends on the maximal angle of the domain because of the regularity [6, 36, 47], which is an interesting topic for future work. Also, for general polygonal domains and Laplace equations it has been shown [6] that

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Gamma)} \leq Ch^{1-\frac{1}{p}},$$

where $p > 2$ depends on the largest angle, and obtaining optimal convergence rates for the $p = 2$ case is another interesting topic for future work.

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Conflict of interest

The author declares no conflict of interest.

References

1. J. Zhu, Q. C. Zeng, A mathematical formulation for optimal control of air pollution, *Sci. China Ser. D*, **46** (2003), 994–1002. https://doi.org/10.1007/BF02959394
2. C. R. A. Martinez, M. E. Vazquez-Mendez, Theoretical and numerical analysis of an optimal control problem related to wastewater treatment, *SIAM J. Control. Optim.*, **38** (2000), 1534–1553. https://doi.org/10.1137/S0363012998345640

3. Y. Bazilevs, T. J. R. Hughes, Weak imposition of Dirichlet boundary conditions in fluid mechanics, *Comput. Fluids*, **36** (2007), 12–26. https://doi.org/10.1016/j.compfluid.2005.07.012

4. T. J. R. Hughes, L. P. Franca, G. M. Hulbert, A new finite element formulation for computational fluid dynamics: VIII. The Galerkin/least squares methods for advecive-diffusive systems, *Comput. Method. Appl. M.*, **73** (1989), 173–189. https://doi.org/10.1016/0045-7825(89)90111-4

5. J. Guzmán, Local analysis of discontinuous Galerkin methods applied to singularly perturbed problems, *J. Numer. Math.*, **14** (2006), 41–56. https://doi.org/10.1515/156939506776382157

6. S. May, R. Rannacher, B. Vexler, Error analysis for a finite element approximation of elliptic Dirichlet boundary control problems, *SIAM J. Control Optim.*, **51** (2013), 2585–2611. https://doi.org/10.1137/080735734

7. B. Ayuso, L. D. Marini, Discontinuous Galerkin methods for advection-diffusion-reaction problems, *SIAM J. Numer. Anal.*, **47** (2009), 1391–1420. https://doi.org/10.1137/080719583

8. D. N. Arnold, F. Brezzi, B. Cockburn, L. D. Marini, Discontinuous Galerkin methods for elliptic problems, In: *Discontinuous Galerkin methods*, Berlin: Springer, 2000. https://doi.org/10.1007/978-3-642-59721-3_5

9. K. Deckelnick, A. Günther, M. Hinze, Finite element approximation of Dirichlet boundary control for elliptic PDEs on two-and three-dimensional curved domains, *SIAM J. Control Optim.*, **48** (2009), 2798–2819. https://doi.org/10.1137/080735369

10. D. Leykekhman, Investigation of commutative properties of discontinuous Galerkin methods in PDE constrained optimal control problems, *J. Sci. Comput.*, **53** (2012), 483–511. https://doi.org/10.1007/s10915-012-9582-y

11. J. Freund, R. Stenberg, On weakly imposed boundary conditions for second order problems, In: *Finite elements in fluids*, Venice, 1995, 327–336.

12. F. Brezzi, G. Manzini, D. Marini, P. Pietra, A. Russo, Discontinuous Galerkin approximations for elliptic problems, *Numer. Meth. Part. D. E.*, **16** (2000), 365–378. https://doi.org/10.1002/1098-2426(200007)16:4¡365::AID-NUM2¿3.0.CO;2-Y

13. P. Neittaanmaki, J. Sprekels, D. Tiba, *Optimization of elliptic systems: Theory and applications*, Springer Science & Business Media, 2006.

14. Z. J. Zhou, N. N. Yan, A survey of numerical methods for convection-diffusion optimal control problems, *J. Numer. Math.*, **22** (2014), 61–85. https://doi.org/10.1515/jnum-2014-0003

15. S. S. Collis, M. Heinkenschloss, *Analysis of the streamline upwind/Petrov Galerkin method applied to the solution of optimal control problems*, CAAM TR02–01, 2002.

16. A. N. Brooks, T. J. R. Hughes, Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations, *Comput. Method. Appl. M.*, **32** (1982), 199–259. https://doi.org/10.1016/0045-7825(82)90071-8

17. G. Chen, J. R. Singler, Y. Zhang, An HDG method for Dirichlet boundary control of convection dominated diffusion PDEs, *SIAM J. Numer. Anal.*, **57** (2019), 1919–1946. https://doi.org/10.1137/18M1208708

18. G. Chen, G. Fu, J. R. Singler, Y. Zhang, A class of embedded DG methods for Dirichlet boundary control of convection diffusion PDEs, *J. Sci. Comput.*, **81** (2019), 623–648. https://doi.org/10.1007/s10915-019-01043-9
19. W. Gong, W. Hu, M. Mateos, J. Singler, X. Zhang, Y. Zhang, A new HDG method for Dirichlet boundary control of convection diffusion PDEs II: Low regularity, *SIAM J. Numer. Anal.*, **56** (2018), 2262–2287. https://doi.org/10.1137/17M1152103
20. W. Hu, M. Mateos, J. R. Singler, Y. Zhang, A new HDG method for Dirichlet boundary control of convection diffusion PDEs I: High regularity, 2018, arXiv: 1801.01461.
21. W. Hu, J. Shen, J. R. Singler, Y. Zhang, X. Zheng, A superconvergent hybridizable discontinuous Galerkin method for Dirichlet boundary control of elliptic PDEs, *Numer. Math.*, **144** (2020), 375–411. https://doi.org/10.1007/s00211-019-01090-2
22. Z. Zhou, X. Yu, N. Yan, Local discontinuous Galerkin approximation of convection-dominated diffusion optimal control problems with control constraints, *Numer. Meth. Part. D. E.*, **30** (2014), 339–360. https://doi.org/10.1002/num.21815
23. E. M. Cliff, M. Heinkenschloss, A. Shenoy, An optimal control problem for flows with discontinuities, *J. Optimiz. Theory App.*, **94** (1997), 273–309. https://doi.org/10.1023/A:1022616327742
24. P. Benner, H. Yücel, *A local discontinuous Galerkin method for Dirichlet boundary control problems*, 2017. https://doi.org/10.13140/RG.2.2.16362.52164
25. E. Burman, P. Hansbo, Edge stabilization for Galerkin approximations of convection-diffusion-reaction problems, *Comput. Method. Appl. M.*, **193** (2004), 1437–1453. https://doi.org/10.1016/j.cma.2003.12.032
26. R. Becker, B. Vexler, Optimal control of the convection-diffusion equation using stabilized finite element methods, *Numer. Math.*, **106** (2007), 349–367. http://dx.doi.org/10.1007/s00211-007-0067-0
27. B. Cockburn, On discontinuous Galerkin methods for convection–dominated problems, In: *Advanced numerical approximation of nonlinear hyperbolic equations*, Berlin: Springer, 1998. http://dx.doi.org/10.1007/BFb0096353
28. D. Leykekhman, M. Heinkenschloss, Local error analysis of discontinuous Galerkin methods for advection-dominated elliptic linear-quadratic optimal control problems, *SIAM J. Numer. Anal.*, **50** (2012), 2012–2038. https://doi.org/10.1137/110826953
29. L. Dedé, A. Quarteroni, Optimal control and numerical adaptivity for advection-diffusion equations, *ESAIM Math. Model. Num.*, **39** (2005), 1019–1040. https://doi.org/10.1051/m2an:2005044
30. M. Heinkenschloss, D. Leykekhman, Local error estimates for SUPG solutions of advection-dominated elliptic linear-quadratic optimal control problems, *SIAM J. Numer. Anal.*, **47** (2010), 4607–4638. https://doi.org/10.1137/090759902
31. E. Casas, J. P. Raymond, Error estimates for the numerical approximation of Dirichlet boundary control for semilinear elliptic equations, *SIAM J. Control. Optim.*, **45** (2006), 1586–1611. https://doi.org/10.1137/050626600

32. L. C. Evans, *Partial differential equations*, Graduate studies in mathematics, 1998.

33. P. Grisvard, *Elliptic problems in nonsmooth domains*, Bost: Pitman Publishing Inc., 1985.

34. J. L. Lions, E. Magenes, *Non-homogeneous boundary value problems and applications*, New York: Springer-Verlag, 1972.

35. F. Tröltzsch, *Optimal control of partial differential equations: Theory, methods, and applications*, American Mathematical Soc., 2010.

36. T. Apel, M. Mateos, J. Pfefferer, A. Rsch, On the regularity of the solutions of Dirichlet optimal control problems in polygonal domains, *SIAM J. Control Optim.*, **53** (2015), 3620–3641. https://doi.org/10.1137/140994186

37. Y. Epshteyn, B. Rivière, Estimation of penalty parameters for symmetric interior penalty galerkin methods, *J. Comput. Appl. Math.*, **206** (2007), 843–872. https://doi.org/10.1016/j.cam.2006.08.029

38. D. N. Arnold, An interior penalty finite element method with discontinuous elements, *SIAM J. Numer. Anal.*, **19** (1982), 742–760. https://doi.org/10.1137/0719052

39. T. Akman, H. Yücel, B. Karasözen, A priori error analysis of the upwind symmetric interior penalty galerkin (sipg) method for the optimal control problems governed by unsteady convection diffusion equations, *Comput. Optim. Appl.*, **57** (2014), 703–729. https://doi.org/10.1007/s10589-013-9601-4

40. A. Cangiani, J. Chapman, E. H. Georgoulis, M. Jensen, On local super-penalization of interior penalty discontinuous galerkin methods, 2012, arXiv:1205.5672.

41. A. Y. Chernyshenko, M. A. Olshanskii, Non-degenerate Eulerian finite element method for solving pdes on surfaces, *Russ. J. Numer. Anal. M.*, **28** (2013), 101–124. https://doi.org/10.1515/rnam-2013-0007

42. M. A. Olshanskii, A. Reusken, Trace finite element methods for PDEs on surfaces, In: *Geometrically unfitted finite element methods and applications*, Springer, 2017. https://doi.org/10.1007/978-3-319-71431-8_7

43. P. G. Ciarlet, *The finite element method for elliptic problems*, Society for Industrial and Applied Mathematics, 2002.

44. A. Quarteroni, S. Quarteroni, *Numerical models for differential problems*, Milan: Springer, 2009.

45. Y. Epshteyn, B. Riviere, Symmetric interior penalty Galerkin method for elliptic problems, 2012.

46. M. Mateos, I. Neitzel, Dirichlet control of elliptic state constrained problems, *Comput. Optim. Appl.*, **63** (2016), 825–853. https://doi.org/10.1007/s10589-015-9784-y

47. T. Apel, M. Mateos, J. Pfefferer, A. Rösch, Error estimates for Dirichlet control problems in polygonal domains: quasi-uniform meshes, *Math. Control Relat. F.*, **8** (2018), 217–245. https://doi.org/10.3934/mcrf.2018010

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