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Comparison of a random model of Hand-Foot-Mouth Disease model with Gaussian and Laplacian parameters

Bir rastgele El-Ayak-Ağız Hastalığı modelinin Normal ve Laplace dağılımına sahip parametrelerle karşılaştırılması

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Abstract
In this study, we investigate the transmission dynamics of Hand-Foot-Mouth disease (HFMD) using a differential equation system with random parameters. We transform the parameters of the existing deterministic model into random variables with Normal and Laplace distributions. Using the results from the simulations of the random model, we analyze the changes in the compartments of the total population. The random model, unlike the deterministic system, enables the analysis of the variations in the transmission dynamics of the disease. Finally, the randomness of the system is interpreted through the comparison of the results from the deterministic and random models.

Keywords: Hand-Foot-Mouth Disease, Laplace distribution, Normal distribution, Random differential equation, Simulation

Öz
Bu çalışmada, El-Ayak-Ağız Hastalığının yayılım dinamikleri bir diferansiyel denklem sistemi ve rastgele parametreler kullanarak incelenmektedir. Var olan deterministik modelin parametreleri Normal dağılım ve Laplace dağılımına sahip rastgele değişkenler haline getirilmektedir. Rastgele modelin simülasyonlarından elde edilen sonuçlarla toplam nüfusun kompartmanlarındaki değişimler analiz edilmektedir. Rastgele model, deterministik sistemin aksine, hastalığın yayılım dinamiklerindeki varyasyonların analizine imkan sağlamaktadır. Son olarak, sistemin rastgele yapısı deterministik ve rastgele sonuçların karşılaştırılması ile yorumlanmaktadır.

Anahtar kelimeler*: El-Ayak-Ağız Hastalığı, Laplace dağılımı, Normal dağılım, Rastgele diferansiyel denklem, Simülasyon*

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1. Introduction
1. Giriş

Compartmental models date back to the pioneering study of W. O. Kermack and A. G. McKendrick in 1927. The SIR model divides the population into three compartments, namely Susceptible, Infected and Recovered elements and monitors the course of the disease through the changes in these compartments. Diseases such as Hepatitis C, Ebola and etc. have been modeled through SIR-based models containing additional compartments and parameters (Merdan et al., 2017; Merdan et al., 2018). One of the diseases that have been modeled using compartmental models is the Hand-Foot-Mouth disease (HFMD).

Hand-Foot-Mouth disease is an infectious disease that is often seen in children. However, adults can also be affected from the disease. Although the disease is mild in most cases, severe symptoms may occur in some cases (World Health Organization, n.d.). Hence, the disease has been studied in detail by researchers, including mathematical modeling studies. Some of the recent modeling studies on HFMD can be listed as follows. Chen et al. have modeled the transmissibility of the disease by using a SIR-based dynamic model and data from China (Chen et al., 2019). Phonchan and Naowarat have made a sensitivity analysis in a SEIQR type model (Phonchan and Naowarat, 2019). Luo et al. have used the SIR model to analyze the interaction of main pathogens that cause HFMD (Luo et al., 2020). Dai et al. have analyzed the spread of the disease in Wenzhou, China using a SEIQRN type model (Dai et al., 2019). Liao et al. have analyzed the spread of the disease between genders with SIR type models for the men and women populations (Liao et al., 2019). Huang et al. have investigated the seasonality of the disease transmission for HFMD using a SEIAR type model (Huang et al., 2019). Chadsuthi and Wichapeng have analyzed the transmission in contaminated environments in Bangkok, Thailand (Chadsuthi and Wichapeng, 2018). Tan and Cao have used a SEIVT type compartmental model to analyze the transmission dynamics of HFMD (Tan and Cao, 2018). Li et al. have used a SEILR type model to investigate the transmission of the disease in mainland China (Li et al., 2019). Pongsumpun and Wongvanich have used SEIR type models for children and grown-ups to model the disease transmission in Thailand (Pongsumpun and Wongvanich, 2018). Shi and Lu have used SEIIRW type model with fractional derivatives to analyze the disease spread (Shi and Lu, 2020). Thus, it is seen that most of the recent models on the transmission of HFMD are SIR and SEIR based compartmental models. Modifications of the SIR model like SEIR model, which contains an extra compartment for exposed individuals, are frequently used modeling various disease transmission dynamics.

In this study, we will use a SEIVT-type compartmental model given by Tan and Cao to analyze the transmission dynamics of HFMD under random effects (Tan and Cao, 2018). The original study is a deterministic study. The system is a SEIR-type based compartmental model with an additional compartment for vaccinated individuals. In our study, we transform the parameters of the deterministic model into random variables to analyze the random transmission dynamics of HFMD. The deterministic model assumes that the parameters are constant quantities, whereas the disease dynamics represented with the parameters can be random in nature. Hence, we will use a random differential equation system to represent this randomness. The motivation of this study is the previous literature on random modeling of several diseases (Merdan et al., 2017; Merdan et al., 2018). Laplace and Normal (Gauss) distributions will be used for the distributions of the random parameters. These two distributions have similar properties, such as being continuous and symmetrical around the mean. The comparison of results from these two distributions and the deterministic case will provide useful insights into the random dynamics of HFMD transmission.

2. The deterministic model of HFMD transmission
2. El Ayak Ağız hastalığının yayılımının deterministik modeli

The compartmental model given by Tan and Cao in 2018 consists of five compartments that divide the total population $N(t)$ (Tan and Cao, 2018).

$$\begin{aligned}
\frac{dS(t)}{dt} &= (1-p)b - \beta S(t)I(t) - (\mu+\omega)S(t) + \\
&\quad \eta_1 V(t) + \eta_2 T(t), \\
\frac{dE(t)}{dt} &= \beta S(t)I(t) - (\mu+\alpha)E(t), \qquad (1) \\
\frac{dI(t)}{dt} &= \alpha E(t) - (\mu+d+\gamma)I(t), \\
\frac{dV(t)}{dt} &= pb - (\mu+\omega+\eta_1)V(t), \\
\frac{dT(t)}{dt} &= \gamma I(t) - (\mu+\omega+\eta_2)T(t).
\end{aligned}$$

Here, $S(t)$ denotes the susceptible population, $E(t)$ denotes the exposed population, $I(t)$ denotes the total infected population, $V(t)$ denotes the vaccinated population and $T(t)$ denotes the recovered population. t is the time variable which denotes the number of days. The deterministic differential equation system (1) shows the changes in the compartments S, E, I, V, T in time. The parameters of the equation system, their descriptions and numerical values are given in Table 1.

Table 1. The parameters of (1) along with their descriptions and numerical values
Tablo 1. *(1) modelinin parametreleri, tanımları ve sayısal değerleri*

Parameter	Description	Value
b	birth rate	2
p	vaccine rate	0.5
β	transmission coefficient	0.04
μ	natural death rate	0.0017
α	progression from latent to infectious	1.75
d	disease induced death rate	0.0034
γ	treatment rate	0.4
ω	removal rate of population	0.125
η_1	loss of immunity rate of vaccinated	0.5
η_2	loss of immunity rate of recovered	0.2

The initial values of system (1) are given as $(\mathrm{S}(0), \mathrm{E}(0), \mathrm{I}(0), \mathrm{V}(0), \mathrm{T}(0)) = (2, 0.7, 0.2, 0.1, 1) \times 10^3$. This set of initial values simulates a population of 2000 susceptible, 700 exposed, 200 infected, 100 vaccinated and 1000 recovered people. The numerical values of the parameters and the initial values have been obtained from the referred study (Tan and Cao, 2018).

3. Random models
3. Rastgele modeller

The deterministic model (1) will be transformed into a system of random differential equations through the use of random parameters with Normal and Laplace distributions. The random parameters with Normal distribution are introduced as follows

$$b^* = b + s_1 Z_1,\ \ p^* = p + s_2 Z_2,\ \ \beta^* = \beta + s_3 Z_3,\ \ \mu^* = \mu + s_4 Z_4,\ \ \omega^* = \omega + s_5 Z_5,$$
$$\eta_1^* = \eta_1 + s_6 Z_6,\ \ \eta_2^* = \eta_2 + s_7 Z_7,\ \ \alpha^* = \alpha + s_8 Z_8,\ \ d^* = d + s_9 Z_9,\ \ \gamma^* = \gamma + s_{10} Z_{10}, \quad (2)$$

where $b, p, \beta, \mu, \omega, \eta_1, \eta_2, \alpha, d, \gamma$ are the deterministic values of the parameters given in Table 1, $\mathrm{s}_i, i = \overline{1,10}$ are the standard deviations of the random parameters and $\mathrm{Z}_i, i = \overline{1,10}$ are independent standard Normally distributed random variables. The standard deviations of the random parameters are assumed to be 5% of their deterministic values for this study. Hence, if the random variables (2) are implemented in the system (1), we obtain the first random model with Normally distributed random effects.

$$\frac{dS(t)}{dt} = \big(1 - (0.5 + 0.025Z_2)\big)(2 + 0.1Z_1) - (0.04 + 0.002Z_3)S(t)I(t) - \big((0.0017 + 0.000085Z_4) + (0.125 + 0.00625Z_5)\big)S(t) + (0.5 + 0.025Z_6)V(t) + (0.2 + 0.01Z_7)T(t),$$

$$\frac{dE(t)}{dt} = (0.04 + 0.002Z_3)S(t)I(t) - ((0.0017 + 0.000085Z_4) + (1.75 + 0.0875Z_8))E(t), \quad (3)$$

$$\frac{dI(t)}{dt} = (1.75 + 0.0875Z_8)E(t) - ((0.0017 + 0.000085Z_4) + (0.0034 + 0.00017Z_9) + (0.4 + 0.02Z_{10}))I(t),$$

$$\frac{dV(t)}{dt} = (0.5 + 0.025Z_2)(2 + 0.1Z_1) - ((0.0017 + 0.000085Z_4) + (0.125 + 0.00625Z_5) + (0.5 + 0.025Z_6))V(t),$$

$$\frac{dT(t)}{dt} = (0.4 + 0.02Z_{10})I(t) - \big((0.0017 + 0.000085Z_4) + (0.125 + 0.00625Z_5) + (0.2 + 0.01Z_7)\big)T(t).$$

In order to obtain a realistic comparison of the cases with Normal and Laplacian random effects, we need to calibrate the Laplacian parameters so that both cases have similar expected values and standard distributions. Firstly, we give the probability density functions of these distributions as an introduction. A random variable X has Normal (Gaussian) distribution if it has the probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right\}, x \in \mathbb{R}. \quad (4)$$

In this case, the random variable has the expected value $E(X) = \mu$ and the variance $Var(X) = \sigma^2$. Standard normal distribution is a special case of Normal distribution where $E(X) = 0$ and $Var(X) = 1$. Similarly, a random variable X has Laplace distribution if it has the probability density function (PDF) (Forbes et al, 2011)

$$f(x) = \frac{1}{2b} exp\left\{-\frac{|x-a|}{b}\right\}, x \in \mathbb{R}. \quad (5)$$

In the case for Laplace distribution, the random variable has the expected value $E(X) = a$ and the variance $Var(X) = 2b^2$. The standard Laplace distribution is a special case of Laplace distribution where $E(X) = 0$ and $Var(X) = 2$. The standard cases of the probability density functions (4) and (5) have been plotted below for a visual comparison (Figure 1).

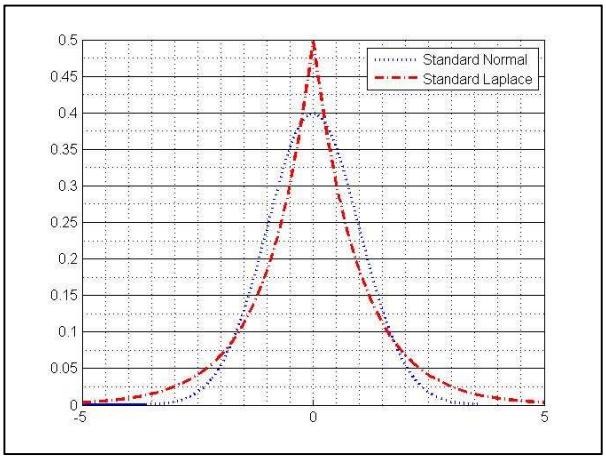

Figure 1. PDF for standard Laplace and standard normal distributions
Şekil 1. *Standart Laplace ve standart Normal dağılımların olasılık yoğunluk fonksiyonları*

The random parameters with Laplace distribution are introduced as follows

$$b^{**} = b + t_1L_1,\ p^{**} = p + t_2L_2,\ \beta^{**} = \beta + t_3L_3,\ \mu^{**} = \mu + t_4L_4,\ \omega^{**} = \omega + t_5L_5, \quad (6)$$

$$\eta_1^{**} = \eta_1 + t_6L_6,\ \eta_2^{**} = \eta_2 + t_7L_7,\ \alpha^{**} = \alpha + t_8L_8,\ d^{**} = d + t_9L_9,\ \gamma^{**} = \gamma + t_{10}L_{10},$$

where $b, p, \beta, \mu, \omega, \eta_1, \eta_2, \alpha, d, \gamma$ are again the deterministic values of the parameters whereas $t_i, i = \overline{1,10}$ are the standard deviations of the random parameters and $L_i, i = \overline{1,10}$ are independent standard Laplacian random variables. The standard deviations of the parameters $b^{**}, p^{**}, \beta^{**}, \mu^{**}, \omega^{**}, \eta_1^{**}, \eta_2^{**}, \alpha^{**}, d^{**}, \gamma^{**}$ are introduced as follows so that they have the same standard deviation as $b^*, p^*, \beta^*, \mu^*, \omega^*, \eta_1^*, \eta_2^*, \alpha^*, d^*, \gamma^*$

$$t_i = \frac{s_i\sqrt{2}}{2}, i = \overline{1,10}. \quad (7)$$

Using these t_i, the random model with Laplacian random effects are obtained as follows:

$$\frac{dS(t)}{dt} = \left(1 - \left(0.5 + \frac{5\sqrt{2}}{200} \times 0.5L_2\right)\right)\left(2 + \frac{5\sqrt{2}}{200} \times 2L_1\right) - (0.04 + \frac{5\sqrt{2}}{200} \times 0.04L_3)S(t)I(t) - \left(\left(0.0017 + \frac{5\sqrt{2}}{200} \times 0.0017L_4\right) + (0.125 + \frac{5\sqrt{2}}{200} \times 0.125L_5)\right)S(t) + (0.5 + \frac{5\sqrt{2}}{200} \times 0.5L_6)V(t) + (0.2 + \frac{5\sqrt{2}}{200} \times 0.2L_7)T(t),$$

$$\frac{dE(t)}{dt} = (0.04 + \frac{5\sqrt{2}}{200} \times 0.04L_3)S(t)I(t) - \left((0.0017 + \frac{5\sqrt{2}}{200} \times 0.0017L_4) + (1.75 + \frac{5\sqrt{2}}{200} \times 1.75L_8)\right)E(t),$$

$$\frac{dI(t)}{dt} = (1.75 + \frac{5\sqrt{2}}{200} \times 1.75L_8)E(t) - \left((0.0017 + \frac{5\sqrt{2}}{200} \times 0.0017L_4) + (0.0034 + \frac{5\sqrt{2}}{200} \times 0.0034L_9) + (0.4 + \frac{5\sqrt{2}}{200} \times 0.4L_{10})\right)I(t), \quad (8)$$

$$\frac{dV(t)}{dt} = (0.5 + \frac{5\sqrt{2}}{200} \times 0.5L_2)(2 + \frac{5\sqrt{2}}{200} \times 2L_1) - \Big((0.0017 + \frac{5\sqrt{2}}{200} \times 0.0017L_4) + (0.125 + \frac{5\sqrt{2}}{200} \times 0.125L_5) + (0.5 + \frac{5\sqrt{2}}{200} \times 0.5L_6)\Big)V(t),$$

$$\frac{dT(t)}{dt} = \Big(0.4 + \frac{5\sqrt{2}}{200} \times 0.4L_{10}\Big)I(t) - \Bigg(\Big(0.0017 + \frac{5\sqrt{2}}{200} \times 0.0017L_4\Big) + \Big(0.125 + \frac{5\sqrt{2}}{200} \times 0.125L_5\Big) + \Big(0.2 + \frac{5\sqrt{2}}{200} \times 0.2L_7\Big)\Bigg)T(t).$$

4. Simulation results
4. Simülasyon sonuçları

The deterministic model (1) has been simulated using the parameter values given in Table 1 and the initial conditions $(S(0), E(0), I(0), V(0), T(0)) = (2, 0.7, 0.2, 0.1, 1) \times 10^3$. The results are shown in the figure below (Figure 2).

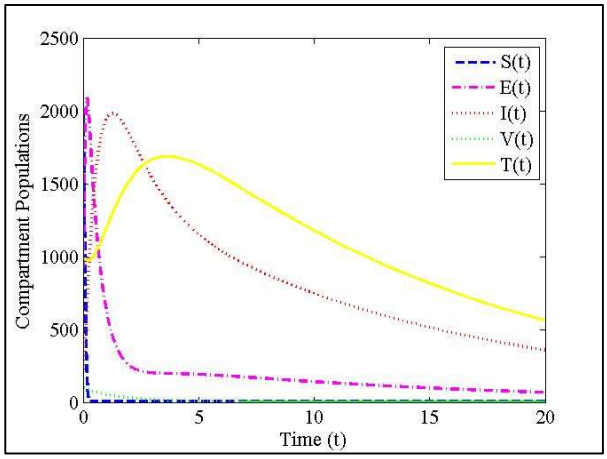

Figure 2. Deterministic results for model (1)
Şekil 2. *(1) modelinin deterministik sonuçları*

The results suggest that the number of susceptible people will assume the minimum value 3.404 at $t = 0.95$ while the maximum value 2000 will be obtained at $t = 0$. The number of exposed people will vary between the minimum value 67.71 obtained at $t = 20$ and the maximum value 2097 obtained at $t = 0.17$. The number of infected people will obtain the minimum value 200 at $t = 0$ and the maximum value 1987 at $t = 1.24$. The number of vaccinated people will obtain the minimum value 1.596 at $t = 20$ and the maximum value 100 at $t = 0$. Lastly, the number of recovered people will assume the minimum value 564.4 at $t = 20$ and the maximum value 1690 at $t = 3.58$.

4.1. Simulation results for Normal parameters
4.1. Normal dağılıma sahip parametreler için simülasyon sonuçları

The random model (3) containing the random parameters (2) with normal distribution has been simulated 5×10^4 times in MATLAB and the following numerical characteristics have been obtained. The minimum and maximum values for the expectations of the model (3) have been given in the following table (Table 2). Additionally, the graphs of the expectations have been given in the figure below (Figure 3).

Table 2. The extremum values for the expectations with Normal parameters
Tablo 2. *Normal dağılıma sahip parametrelerle beklenen değerlerin uç değerleri*

	Minimum (Time)	**Maximum (Time)**
$E(S(t))$	3.414 (0.94)	2000 (0)
$E(E(t))$	68.05 (20)	2094 (0.17)
$E(I(t))$	200 (0)	1985 (1.28)
$E(V(t))$	1.597 (20)	100 (0)
$E(T(t))$	565.5 (20)	1688 (3.6)

The comparison between the deterministic results given above and the extremum values for the expected values given in Table 2 show the similarity of the results for both cases. It is obvious from this comparison that the random model (3) with Normally distributed random effects is perfectly capable of modeling the disease transmission dynamics for Hand-Foot-Mouth disease (HFMD).

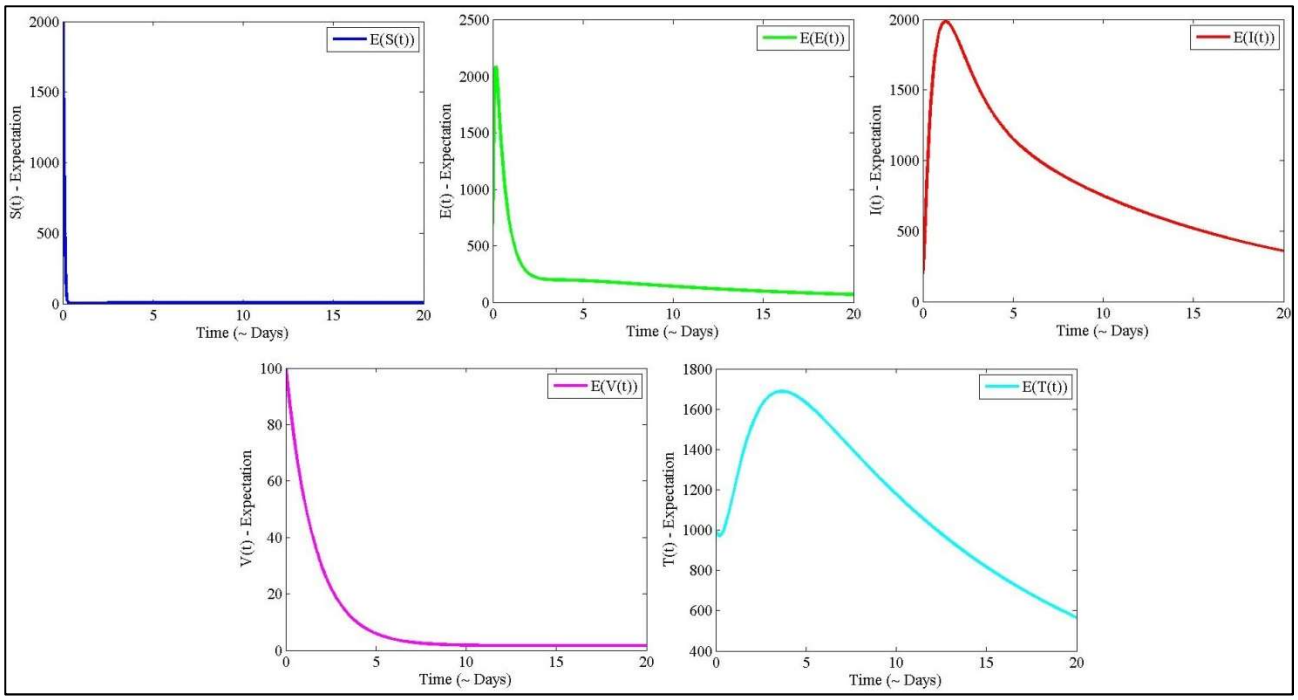

Figure 3. Expected values for model (3)
Şekil 3. *(3) modeli için beklenen değerler*

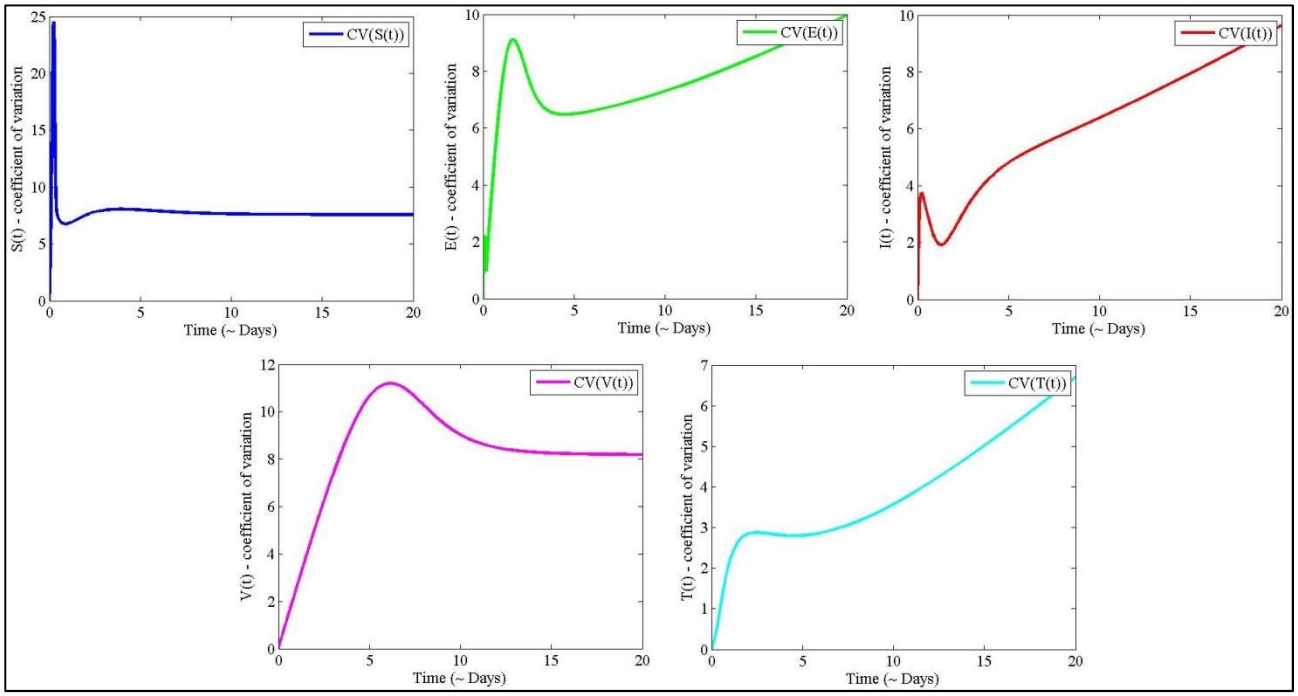

Figure 4. Coefficients of variation for model (3)
Şekil 4. *(3) modeli için değişim katsayıları*

The results for the coefficients of variation are given as below (Table 3 and Figure 4).

Table 3. The minimum and maximum values for the coefficients of variation
Tablo 3. *Değişim katsayıları için minimum ve maksimum değerler*

	Minimum (Time)	Maximum (Time)
$CV(S(t))$	0 (0)	24.55 (0.24)
$CV(E(t))$	0 (0)	9.975 (20)
$CV(I(t))$	0 (0)	9.634 (20)
$CV(V(t))$	0 (0)	11.19 (6.01)
$CV(T(t))$	0 (0)	6.693 (20)

Note that the coefficient of variation is given as $100 \times (Standard\ Deviation)/(Expected\ Value)$. The results for the confidence intervals of the expectations are given below (Figure 5). Here, three standard deviations are used to obtain the confidence intervals and the dashed line shows the upper end of the confidence interval whereas the dash-dot lines are the lower ends of the interval. The extremum values within the confidence intervals are given in the table below (Table 4).

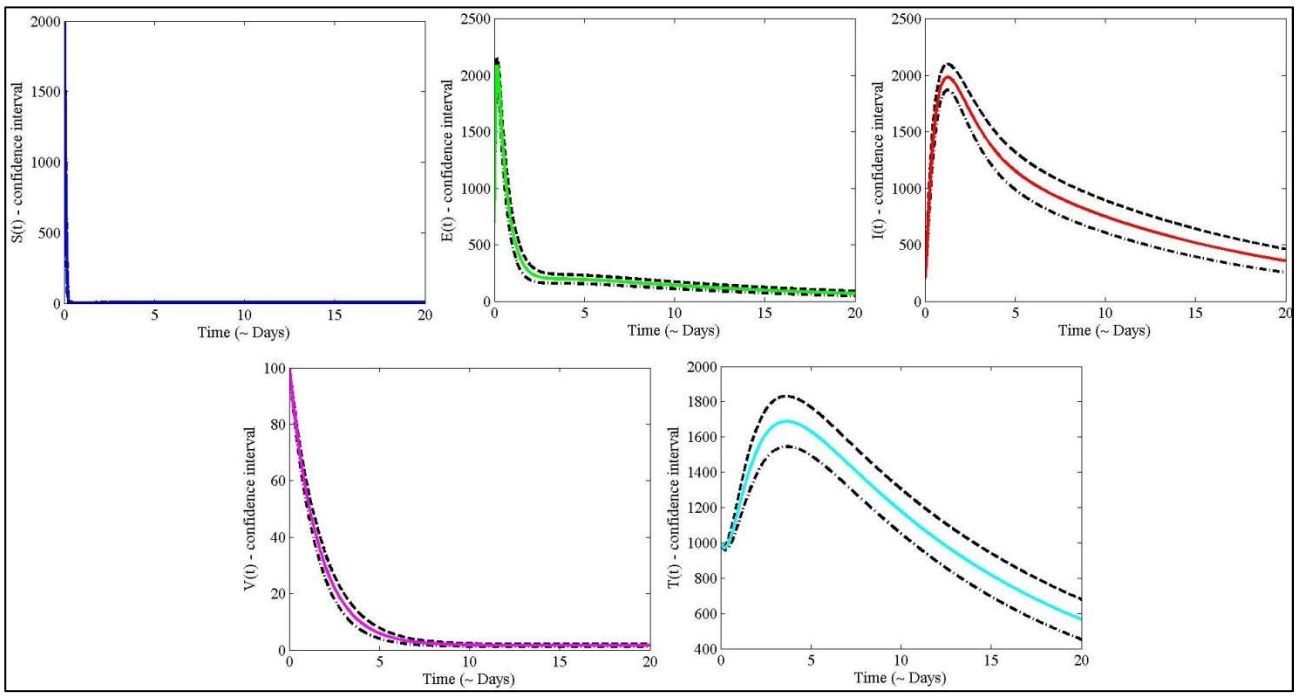

Figure 5. Confidence intervals for the expectations of model (3)
Şekil 5. *(3) modelinin beklenen değerleri için güven aralıkları*

Table 4. The extremum values for the expectations within confidence intervals
Tablo 4. *Beklenen değerlerin güven aralıkları içindeki uç değerleri*

	Minimum (Time)	Maximum (Time)
$E(S(t)) \pm 3std(S(t))$	2.722 (0.95)	2000 (0)
$E(E(t)) \pm 3std(E(t))$	47.68 (20)	2156 (0.17)
$E(I(t)) \pm 3std(I(t))$	200 (0)	2099 (1.23)
$E(V(t)) \pm 3std(V(t))$	1.205 (16.95)	100 (0)
$E(T(t)) \pm 3std(T(t))$	452 (20)	1831 (3.54)

4.2. Simulation results for Laplacian parameters
4.2. Laplace dağılımına sahip parametreler için simülasyon sonuçları

The random model (8) contains Laplacian parameters that have been calibrated to have similar expected values and variances to the case with Normal parameters. The model has been simulated in MATLAB 5×10^4 times. The minimum and maximum values obtained for the expected values of the compartments have been given in Table 5.

Table 5. The minimum and maximum values for the expectations with Laplacian parameters
Tablo 5*. Laplace dağılımına sahip parametrelerle beklenen değerlerin minimum ve maksimum değerleri*

	Minimum (Time)	Maximum (Time)
$E(S(t))$	3.414 (0.93)	2000 (0)
$E(E(t))$	68.1 (20)	2094 (0.17)
$E(I(t))$	200 (0)	1985 (1.25)
$E(V(t))$	1.598 (20)	100 (0)
$E(T(t))$	565.9 (20)	1688 (3.5)

It can be seen that the results for the extremum values are obtained similarly to the results for the case with Normal parameters. The similarity between the results of the case with Laplacian and Normal parameters can also be seen in the figure below (Figure 6). The results show that the random behaviors of the compartments with Laplacian random effects are in correspondence to the deterministic results as well.

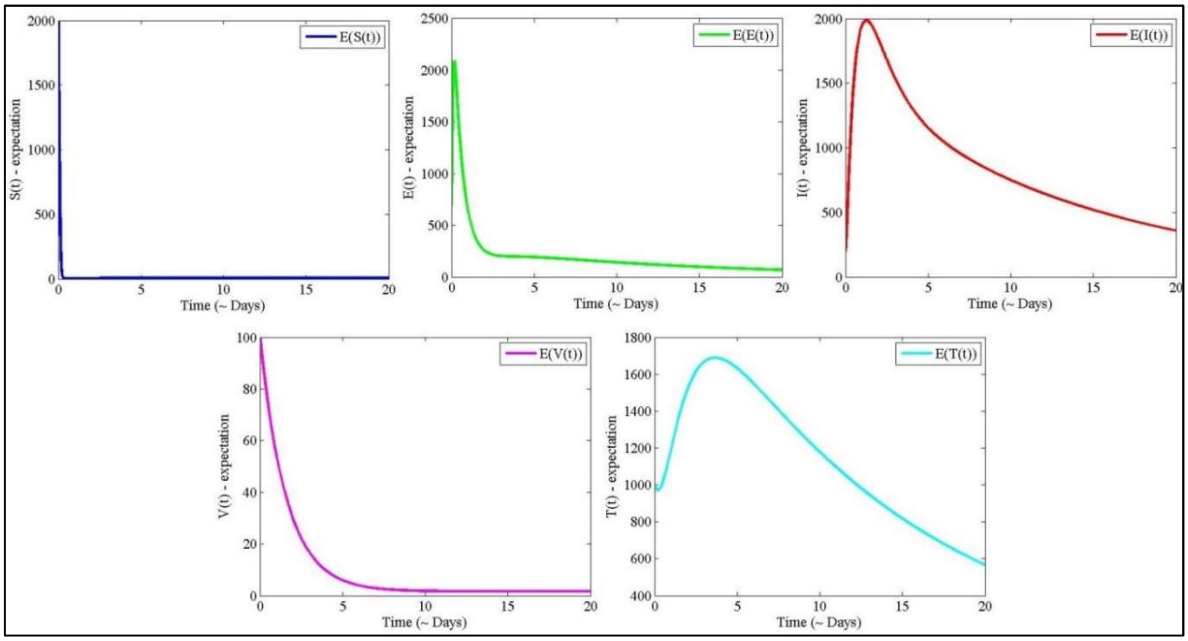

Figure 6. Expected values for model (8)
Şekil 6. *(8) modelinin beklenen değerleri*

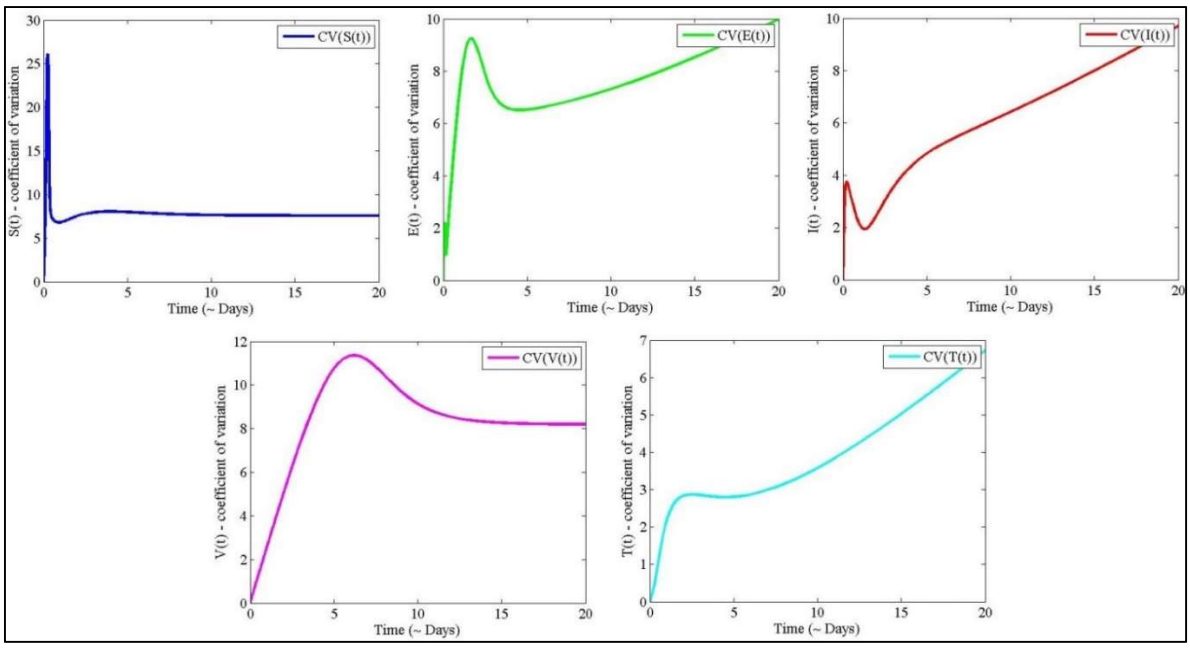

Figure 7. Coefficients of variation for model (8)
Şekil 7. *(8) modeli için değişim katsayıları*

The minimum and maximum values for the coefficients of variation (Figure 7) for the random model with Laplacian parameters have been given in the table below (Table 6).

Figure 7 and Table 6 shows that, just like the case for expected values, similar results have been obtained for the coefficients of variation.

Table 6. The minimum and maximum values for the coefficients of variation
Tablo 6. *Değişim katsayıları için minimum ve maksimum değerler*

	Minimum (Time)	Maximum (Time)
$CV(S(t))$	0 (0)	26.12 (0.25)
$CV(E(t))$	0 (0)	9.987 (20)
$CV(I(t))$	0 (0)	9.704 (20)
$CV(V(t))$	0 (0)	11.36 (6.09)
$CV(T(t))$	0 (0)	6.721 (20)

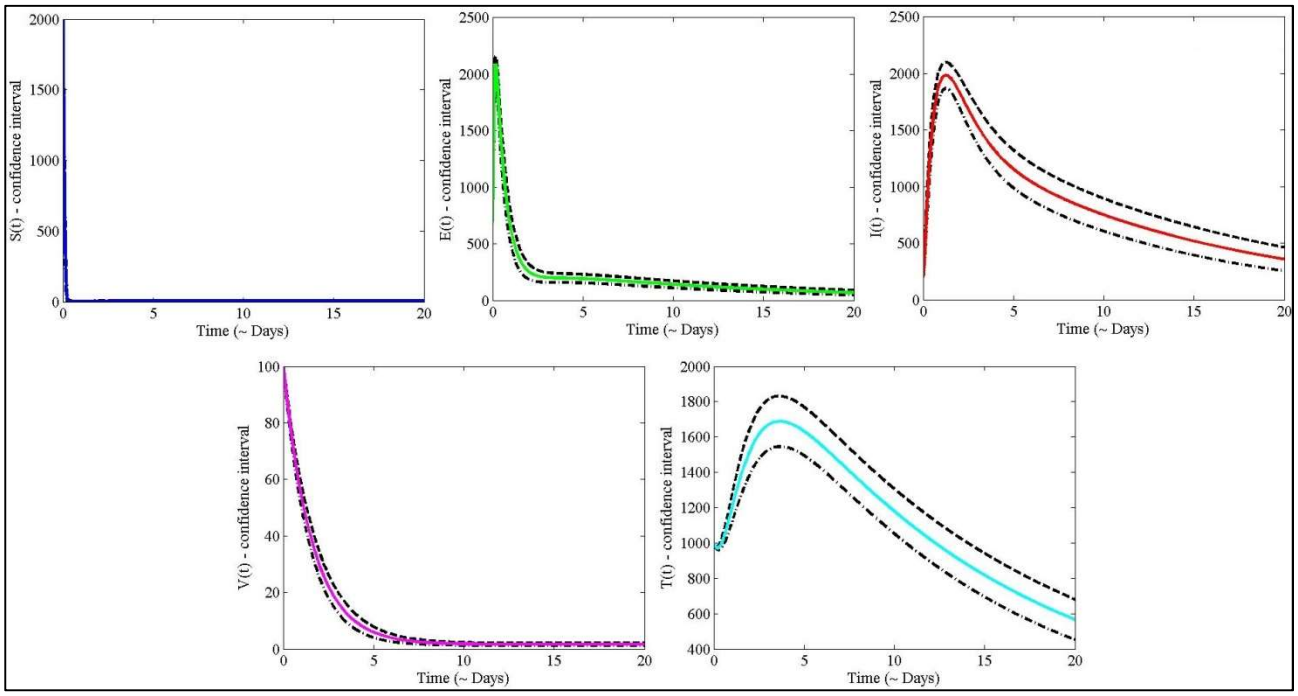

Figure 8. Confidence intervals for the expectations of model (8)
Şekil 8. *(8) modelinin beklenen değerleri için güven aralıkları*

Table 7. The extremum values for the expectations within confidence intervals
Tablo 7. *Beklenen değerlerin güven aralıkları içindeki uç değerleri*

	Minimum (Time)	Maximum (Time)
$E(S(t)) \pm 3std(S(t))$	2.718 (0.94)	2000 (0)
$E(E(t)) \pm 3std(E(t))$	47.7 (20)	2156 (0.17)
$E(I(t)) \pm 3std(I(t))$	200 (0)	2100 (1.22)
$E(V(t)) \pm 3std(V(t))$	1.205 (17.9)	100 (0)
$E(T(t)) \pm 3std(T(t))$	451.8 (20)	1831 (3.5)

The results for the confidence intervals of the expectations for the case with Laplacian parameters are given below (Figure 8). Extremum values of the expectations within the confidence intervals are given in the table below (Table 7).

The similarity can be observed for the cases with Normal and Laplacian parameters here too.

4.3. Comparison of deterministic and random cases
4.3. Deterministik ve rastgele durumların karşılaştırılması

Results for the deterministic case given in Figure 2, the random case with Normal random effects given in Figure 3 and Table 2, and the random case with Laplacian random effects given in Figure 6 and Table 5 can be investigated further to show that the random models (3) and (8) are perfectly capable of

modeling the transmission dynamics of HFMD. Comparison of the deterministic extremum values for the compartments and the extremum values for the random expectations are given in the table below (Table 8).

Table 8. The extremum values for all cases
Tablo 8. *Tüm durumlar için uç değerler*

	Minimum			Maximum		
	Deterministic results (Time)	**Normal results (Time)**	**Laplacian results (Time)**	**Deterministic results (Time)**	**Normal results (Time)**	**Laplacian results (Time)**
$S(t)$	3.404 (0.95)	3.414 (0.94)	3.414 (0.93)	2000 (0)	2000 (0)	2000 (0)
$E(t)$	67.71 (20)	68.05 (20)	68.1 (20)	2097 (0.17)	2094 (0.17)	2094 (0.17)
$I(t)$	200 (0)	200 (0)	200 (0)	1987 (1.28)	1985 (1.28)	1985 (1.25)
$V(t)$	1.596 (20)	1.597 (20)	1.598 (20)	100 (0)	100 (0)	100 (0)
$T(t)$	564.4 (20)	565.5 (20)	565.9 (20)	1690 (3.58)	1688 (3.6)	1688 (3.5)

The behaviors of the compartments are obtained similarly for all three cases with similar extremum values as seen in the table above (Table 8). Note that the random results are for the expectations.

Coefficients of variation (CV) show that the compartment $S(t)$ has the highest randomness with a maximum CV of 26.12% for Laplacian parameters and 24.55% for Normal parameters. Note that the random parameters (2) and (6) have 5% CV because of their random definitions. This is because the standard deviations of the random parameters have been assigned a value that equals to 5% of their deterministic values and hence their expectations. The CV for the compartments $E(t)$ and $I(t)$ obtain a maximum value of almost 10% for both cases. The maximum value of the CV for $V(t)$ is obtained around 11%, whereas the maximum CV for $T(t)$ is obtained around 6.7% for both cases.

The models (1), (3) and (8) show that the number of susceptible people will decrease rapidly in the beginning of the process and maintain a level close to zero until the end of the process. The number of exposed people gets its peak value in the beginning of the process and decreases until the end. Similarly, the number of infected and recovered people increases in the beginning and start decreasing once they have obtained their maximum values. The number of vaccinated people decreases through the process.

5. Conclusion
5. Sonuç

In this study, a deterministic model of Hand-Foot-Mouth disease (HFMD) transmission has been analyzed under Normal and Laplacian random effects. The random effects with Normal and Laplacian distributions have been calibrated to obtain the same expected value and standard deviation for the random parameters. This approach enables an accurate comparison of the results for the deterministic and random cases. The comparison shows that the random models (3) and (8) give expected values similar to the deterministic results obtained from the model (1). This means that the models under random effects are capable of modeling the disease transmission dynamics. In addition to the expected values, the random models enable the analysis of other numerical characteristics of the results. Results for variations, standard deviations, coefficients of variation and confidence intervals for expected values enable the analysis of the changes in the deterministic results. This analysis cannot be done by using the deterministic model. Hence, the random model offers much more than the random model does. Note that this approach of obtaining random models through the use of random effects can be generalized to any compartmental model used for analyzing various disease dynamics.

Author contribution
Yazar katkısı

All authors contributed to the manuscript equally. All authors have read and approved the final manuscript.

Declaration of ethical code
Etik beyanı

The authors of this article declare that the materials and methods used in this study do not require ethical committee approval and/or legal-specific permission.

Conflicts of interest
Çıkar çatışması beyanı

The authors declare no conflict of interest.

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