Research Article

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On the continuity in *q* of the family of the limit *q*-Durrmeyer operators

https://doi.org/10.1515/dema-2023-0157
received August 9, 2023; accepted January 23, 2024

Abstract: This study deals with the one-parameter family $\{D_q\}_{q\in[0,1]}$ of Bernstein-type operators introduced by Gupta and called the limit q-Durrmeyer operators. The continuity of this family with respect to the parameter q is examined in two most important topologies of the operator theory, namely, the strong and uniform operator topologies. It is proved that $\{D_q\}_{q\in[0,1]}$ is continuous in the strong operator topology for all $q \in [0, 1]$. When it comes to the uniform operator topology, the continuity is preserved solely at $q = 0$ and fails at all $q \in (0, 1]$. In addition, a few estimates for the distance between two limit q-Durrmeyer operators have been derived in the operator norm on $C[0, 1]$.

Keywords: q-Durrmeyer operator, q-Bernstein operator, operator norm, strong operator topology, uniform operator topology

MSC 2020: 47B38, 41A36

1 Introduction

Among numerous modifications of the Bernstein polynomials, a special place is occupied first by the Kantorovich and later by the Durrmeyer polynomials, both of which serve to approximate not only continuous but also integrable functions. These polynomials had attracted interest of many researchers, and the intensive work in the area is still going on [1–7].

Along with the advancement of q-calculus, the popularity of the aforementioned operators inspired the study of q-analogues for these operators. See, for example, [8–10] for the results on the q-analogues of the Kantorovich operators.

As for the Durrmeyer operators, currently, there are several q-versions. The first q-analogue of the Durrmeyer operator was introduced by Derriennic [11] in 2005. Thereafter, the q-analogues due to Gupta and Wang [12,13] emerged (see also [8, Chapter 4]). In the present work, the spotlight is on the limit q-Durrmeyer operator D_q as defined in [12]. To be more precise, Gupta considered a q-analogue of the Durrmeyer operator, $D_{n,q} : C[0, 1] \to \mathcal{P}_n$ as polynomials with respect to the q-Bernstein basis possessing q-integral coefficients and proved that, for each $f \in C[0, 1]$,

$$(D_{n,q} f)(x) \to (D_q f)(x), \quad \text{uniformly on } [0, 1],$$

where $D_q f = D_{\infty,q} f$ denotes the limit q-Durrmeyer operator introduced as follows.

Definition 1.1. [12] Given $q \in (0, 1)$, the *limit q-Durrmeyer operator* $D_q : C[0, 1] \to C[0, 1]$ is given by:

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$$(D_q f)(x) = (D_{\infty,q} f)(x) = \begin{cases} \sum_{k=0}^{\infty} A_{k,q}(f) p_k(q;\, x), & x \in [0, 1), \\ f(1), & x = 1, \end{cases} \tag{1.1}$$

where

$$A_{k,q}(f) = (q^{k+1};\, q)_\infty \sum_{j=0}^{\infty} \frac{f(q^j) q^{j(k+1)}}{(q;\, q)_j} \tag{1.2}$$

and

$$p_k(q;\, x) = \frac{(x;\, q)_\infty\, x^k}{(q;\, q)_k}, \quad k = 0, 1, \ldots. \tag{1.3}$$

To extend this definition for $q \in [0, 1]$, it is natural to set $(D_0 f)(x) := f(1)$ and $(D_1 f)(x) := f(x)$.

Remark 1.1. In [12], the coefficients $A_{k,q}(f)$ are presented in the form of the Jackson q-integral.

Remark 1.2. It has to be emphasized that $D_q f = f$ if and only if f is a constant function, which is drastically in contrast with the classical case of $q = 1$, *i.e.*, with the Durrmeyer operator, for which

$$(D_n f)(x) \to f(x), \quad \text{uniformly on } [0, 1] \quad \text{whatever } f \in C[0, 1] \text{ is}.$$

Remark 1.3. The functions $p_k(q;\, x)$ occur as block functions in the construction of the limit q-Bernstein operator B_q [14]. The latter comes out as the limit for a sequence of the q-Bernstein polynomials defined by Phillips in [15], and also as the limit for a sequence of the q-Meyer-König and Zeller operators considered in [16].

Definition 1.2. [14] For $q \in (0, 1)$, the *limit q-Bernstein operator* $B_q : C[0, 1] \to C[0, 1]$ is given by:

$$(B_q f)(x) = \begin{cases} \sum_{k=0}^{\infty} f(1 - q^k) p_k(q;\, x), & x \in [0, 1), \\ f(1), & x = 1, \end{cases} \tag{1.4}$$

where $p_k(q;\, x)$ are as in (1.3). In addition, $(B_0 f)(x) := f(0)(1 - x) + f(1)x$ and $(B_1 f)(x) := f(x)$.

Remark 1.4. Obviously, all the block functions p_k are non-negative on [0,1]. Furthermore, by virtue of the Euler identity (see, e.g. [17, Chapter 10, Corollary 10.2.2]):

$$\frac{1}{(x;\, q)_\infty} = \sum_{k=0}^{\infty} \frac{x^k}{(q;\, q)_k}, \quad |x| < 1, \; |q| < 1, \tag{1.5}$$

one derives that $\sum_{k=0}^{\infty} p_k(q;\, x) \equiv 1$ for $x \in [0, 1)$, whence

$$||B_q|| = ||D_q|| = 1. \tag{1.6}$$

It can be observed that operators (1.1) and (1.4) differ in the coefficients of the block functions (1.3), the fact leading to the dissimilarity in their properties. In what follows, a new relation between the two operators is going to be derived in Lemma 2.1. The properties of B_q have been studied in various aspects revealing connections with other areas. We refer to [18–20]. Meanwhile, the properties of D_q hitherto have not been investigated widely. This study takes action in this direction.

In this work, the continuity of the family $\{D_q\}_{q\in[0,1]}$ with respect to the parameter q is examined. It is proved that $\{D_q\}_{q\in[0,1]}$ is continuous in the strong operator topology for all $q \in [0, 1]$, while, in the uniform operator topology, the continuity holds only at $q = 0$.

This article is organized as follows: in Section 2, a new relation between operators B_q and D_q has been derived and applied to prove the continuity of $\{D_q\}_{q\in[0,1]}$ in the strong operator topology. Section 3 contains the results on the uniform operator topology together with estimates for the distance between two q-Durrmeyer operators. Conclusions are presented in Section 4.

2 Continuity in the strong operator topology

Prior to presenting the results of this section, let us recall the needed notions and terminology. Throughout the text, $C[0,1]$ is assumed to be equipped with the maximum modulus norm, while $L(C[0,1])$ stands for the space of bounded linear operators on $C[0,1]$. A family of operators $\{T_s\}_{s\in[a,b]}\subset L(C[0,1])$ is said to be *continuous in the strong operator topology* if, for each $f\in C[0,1]$, the map $s\mapsto T_s f$ is continuous as a map $[a,b]\to C[0,1]$. In other words, $\{T_s\}_{s\in[a,b]}$ is continuous in the strong operator topology if, given $r\in[a,b]$, the equality

$$\lim_{s\to r}\|T_s f - T_r f\| = 0 \tag{2.1}$$

holds for each $f\in C[0,1]$. Clearly, to prove (2.1), it suffices to show that, given $r\in[a,b]$ and $f\in C[0,1]$, the sequence $\{\|T_{s_n} f - T_r f\|\}\to 0$ whatever $\{s_n\}\to r$ is.

The continuity in q for several q-analogues of the Bernstein operator has already been studied (see, for example, [21,22]). The following lemma plays a key role in proving the continuity of $\{D_q\}_{q\in[0,1]}$ due to the fact that it establishes a connection between operators (1.1) and (1.4).

Lemma 2.1. *Given $q\in(0,1)$ and $f\in C[0,1]$, let $g_q(x) = (q(1-x);q)_\infty \rho_q(q(1-x))$, where*

$$\rho_q(x) = \sum_{j=0}^{\infty}\frac{f(q^j)}{(q;q)_j}x^j. \tag{2.2}$$

Then, $D_q f\equiv B_q g_q$.

Proof. Starting from (1.2), one arrives at

$$A_{k,q}(f) = (q^{k+1};q)_\infty\sum_{j=0}^{\infty}\frac{f(q^j)q^{(k+1)j}}{(q;q)_j} = [(qx;q)_\infty\,\rho_q(qx)]\Big|_{x=q^k} =: h(q^k).$$

Set $g_q(x) = h(1-x)$. Then, one has $A_{k,q}(f) = h(q^k) = g_q(1-q^k)$. Therefore,

$$(D_q f)(x) = \sum_{k=0}^{\infty}h(q^k)p_k(q;x) = \sum_{k=0}^{\infty}g_q(1-q^k)p_k(q;x) = (B_q g_q)(x). \qquad \square$$

Observe that, for a settled function f, its ancillary function g_q depends not solely on f, but also on q. Consequently, when considering a sequence $\{D_{q_n} f\}$, one has to deal with $\{B_{q_n} g_{q_n}\}$. The next lemma serves the purpose.

Lemma 2.2. *Let $\rho_q(x)$ be given by* (2.2). *If $\{q_n\}\to r\in(0,1)$, then $\{\rho_{q_n}(q_n x)\}\to\rho_r(rx)$ uniformly on* $[0,1]$.

Proof. Without loss of generality, it can be assumed that $0<\alpha\leqslant q_n, r\leqslant\beta<1$. Clearly, given $\varepsilon>0$, there is $M_0\in\mathbb{N}$ such that

$$\sum_{j=M_0+1}^{\infty}\left|\frac{f(q^j)}{(q;q)_j}(qx)^j\right| < \varepsilon, \quad \text{for all } q\in[\alpha,\beta],\ x\in[0,1].$$

Indeed, for M_0 large enough,

$$\sum_{j=M_0+1}^{\infty}\left|\frac{f(q^j)}{(q;\,q)_j}(qx)^j\right| \leqslant \frac{\|f\|}{(\alpha;\,\alpha)_\infty}\sum_{j=M_0+1}^{\infty}\beta^j = \frac{\|f\|}{(\alpha;\alpha)_\infty}\frac{\beta^{M_0+1}}{1-\beta} < \varepsilon.$$

Hence,

$$|\rho_{q_n}(q_n x) - \rho_r(rx)| \leqslant \left|\sum_{j=0}^{M_0}\frac{f(q_n^j)}{(q_n;\,q_n)_j}(q_n x)^j - \sum_{j=0}^{M_0}\frac{f(r^j)}{(r;\,r)_j}(rx)^j\right| + 2\varepsilon =: |P_{q_n}(x) - P_r(x)| + 2\varepsilon.$$

Since $\{P_{q_n}\}$ is a sequence of polynomials of degree at most M_0 and $\{P_{q_n}\} \to P_r$ coefficient-wise, the sequence converges uniformly on $[0,1]$, i.e., $|P_{q_n}(x) - P_r(x)| < \varepsilon$ for all $x \in [0,1]$ and n large enough. Thus, $|\rho_{q_n}(q_n x) - \rho_r(rx)| < 3\varepsilon$, $x \in [0,1]$ for n large enough. □

It was shown in [21, Corollary 4] that if $\{q_n\} \to r \in (0,1)$, then $\{(q_n x;\, q_n)_\infty\} \to (rx;\, r)_\infty$ as $n \to \infty$ uniformly on $[0,1]$. Together with Lemma 2.2, this leads to:

Corollary 2.3. *If* $\{q_n\} \to r \in (0,1)$, *then* $\{g_{q_n}(x)\} \to g_r(x)$ *uniformly on* $[0,1]$.

Here comes the main result of this section.

Theorem 2.4. *The family of operators* $\{D_q\}$ *is continuous with respect to* q *in the strong operator topology for all* $q \in [0,1]$, *i.e., for every* $f \in C[0,1]$, *the sequence* $\{(D_{q_n} f)(x)\} \to (D_r f)(x)$, *uniformly on* $[0,1]$ *whenever* $\{q_n\} \to r \in [0,1]$.

Proof. (i) First, let $r \in (0,1)$. The continuity of $\{D_q\}_{q\in[0,1]}$ at r will be proved using the following result presented in Theorem 1 of [21]:

The family $\{B_q\}_{q\in[0,1]}$ *of the limit q-Bernstein operators is continuous in the strong operator topology for all* $q \in [0,1]$.

Note that since the function g_q depends not only on f, but also on q, the cited theorem cannot be applied directly.

Now, given $f \in C[0,1]$ and $\{q_n\} \to r$, consider

$$\begin{aligned}\|D_{q_n} f - D_r f\| = \|B_{q_n} g_{q_n} - B_r g_r\| &\leqslant \|B_{q_n} g_{q_n} - B_{q_n} g_r\| + \|B_{q_n} g_r - B_r g_r\| \\ &\leqslant \|B_{q_n}\|\cdot\|g_{q_n} - g_r\| + \|B_{q_n} g_r - B_r g_r\|.\end{aligned}$$

At this point, the fact that $\|B_{q_n}\| = 1$ together with Corollary 2.3 implies that the first term in the latter inequality tends to 0 as $n \to \infty$. The second term tends to 0 as $n \to \infty$, due to the theorem cited earlier. Thus, one derives that $\{D_q\}$ is continuous in the strong operator topology for $q \in (0,1)$.

(ii) Let $\{q_n\} \to 0$. As $(D_{q_n} f)(1) = f(1)$, it suffices to examine only the case $x \in [0,1)$. By virtue of Remark 1.4,

$$\|(D_{q_n} f)(x) - f(1)\| \leqslant \sum_{k=0}^{\infty}|A_{k,q_n}(f) - f(1)|p_{k,q_n}(x) \leqslant \max_k |A_{k,q_n}(f) - f(1)|.$$

To estimate the latter maximum, note that since $(q_n;\, q_n)_\infty \leqslant (q_n^{k+1};\, q_n)_\infty \leqslant 1$, one has

$$0 \leqslant 1 - (q_n^{k+1};\, q_n)_\infty \leqslant 1 - (q_n;\, q_n)_\infty, \quad k = 0, 1, \ldots$$

Taking into account the known inequality,

$$1 - \prod_{j=1}^{m}(1 - a_j) \leqslant a_1 + a_2 + \cdots + a_m, \quad a_i \in (0,1),$$

and passing to the limit as $m \to \infty$, one arrives at

$$1 - (q_n; q_n)_\infty \leqslant \sum_{j=1}^{\infty} q_n^j = \frac{q_n}{1 - q_n} \leqslant 2q_n, \quad \text{for } q_n \leqslant \frac{1}{2}.$$

Therefore,

$$|A_{k,q_n}(f) - f(1)| = \left| (q_n^{k+1}; q_n)_\infty f(1) + (q_n^{k+1}; q_n)_\infty \sum_{j=1}^{\infty} \frac{f(q_n^j) q_n^{j(k+1)}}{(q_n; q_n)_j} - f(1) \right|$$
$$\leqslant |f(1)|[1 - (q_n; q_n)_\infty] + \frac{\|f\|}{(1/2;1/2)_\infty} \frac{q_n^{k+1}}{1 - q_n} \leqslant \text{Const} \cdot q_n,$$

when $0 < q_n \leqslant 1/2$ regardless of k. Hence, $\max_k |A_{k,q_n}(f) - f(1)| \to 0$ as $q_n \to 0$. This shows that when $\{q_n\} \to 0$, $\|D_{q_n} f - D_0 f\| \to 0$ for any $f \in C[0,1]$. Thus, $\{D_q\}$ is continuous in the strong operator topology at $q = 0$.

(*iii*) Let $\{q_n\} \to 1$. The proof of this part is based on Korovkin's theorem and it is given by V. Gupta in [12, Theorem 5]. For the convenience of the reader, its details are provided here. Since $\{D_q\}$ is a family of positive linear operators on $C[0,1]$, the uniform convergence $\{(D_q f)(x)\} \to f(x)$ as $q \to 1$ has to be demonstrated only for the test functions $e_0 = 1$, $e_1 = x$ and $e_2 = x^2$. Using the identities $(D_q e_0)(x) = 1$, $(D_q e_1)(x) = qx + 1 - q$, and $(D_q e_2)(x) = q^4 x^2 + (1 - q)q(1+q)^2 x + (1 - q)^2(1+q)$, one derives that, when $q \to 1$,

$$(D_q e_k)(x) \to e_k(x), \quad \text{uniformly on } [0,1] \quad \text{for } k = 0, 1, 2.$$

As a result, Korovkin's theorem yields that $\{D_q\}$ is continuous in the strong operator topology at $q = 1$.

The theorem is proved. □

3 Lack of continuity in the uniform operator topology

In the previous section, it has been demonstrated that the family $\{D_q\}_{q\in[0,1]}$ is continuous with respect to q in the strong operator topology. It turns out that, in the uniform operator topology, the situation is radically diverse.

To begin with, let us recollect that a family $\{T_s\}_{s\in[a,b]} \subset L(C[0,1])$ is said to be *continuous in the uniform operator topology* at a point $r \in [a,b]$ if the map $s \mapsto T_s$ is continuous as a map from $[a,b]$ into $L(C[0,1])$ at $s = r$ with respect to the operator norm on $C[0,1]$, i.e., if $\lim_{s\to r}\|T_s - T_r\| = 0$. It is going to be proved that, as opposed to the circumstances associated with the strong operator topology, the family $\{D_q\}_{q\in[0,1]}$ is discontinuous at every point of $[0,1]$ except for $q = 0$.

For the sequel, the following notation comes in handy. Given $a \in [-1,1]$ and $q \in (0,1)$, define functions $\hat{f}_a, \check{f}_a \in C[0,1]$ as:

$$\hat{f}_a(x) = \begin{cases} 1, & \text{if } x \leqslant q \\ a, & \text{if } x = 1 \end{cases} \quad \text{and} \quad \check{f}_a(x) = \begin{cases} -1, & \text{if } x \leqslant q \\ a, & \text{if } x = 1. \end{cases} \tag{3.1}$$

It should be emphasized that the behavior of these functions on the interval $(q,1)$ is of no importance due to the fact that it does not change their images under D_q. In general, it has to be pointed out that the image $D_q f$ depends only on the values of f on the time scale:

$$\mathbb{J}_q := \{0\} \cup \{q^j\}_{j=0}^{\infty}. \tag{3.2}$$

The next observation is essential to prove the main finding of this section.

Lemma 3.1. *Let* $\hat{f}_a, \check{f}_a \in C[0,1]$ *be as in* (3.1). *Then,* $(D_q \hat{f}_a)(x)$ *is a decreasing function on* $[0,1]$, *while* $(D_q \check{f}_a)(x)$ *is an increasing function on* $[0,1]$.

Proof. Note that, by the definition, $(D_q\hat{f}_a)(1) = (D_q\check{f}_a)(1) = a$. In order to apply Lemma 2.1, one proceeds as follows: by means of Euler's identity (1.5),

$$\hat{p}_q(x) = \sum_{j=0}^{\infty} \frac{\hat{f}(q^j)x^j}{(q;\,q)_j} = a + \sum_{j=1}^{\infty} \frac{x^j}{(q;\,q)_j} = a - 1 + \frac{1}{(x;\,q)_\infty},$$

whence,

$$\hat{g}_q(x) = 1 - (1-a)(q(1-x);\,q)_\infty.$$

Likewise,

$$\check{g}_q(x) = -1 + (1+a)(q(1-x);\,q)_\infty.$$

By virtue of Lemma 2.1, one obtains:

$$(D_q\hat{f}_a)(x) = (B_q\hat{g}_q)(x) \quad \text{and} \quad (D_q\check{f}_a)(x) = (B_q\check{g}_q)(x),$$

where B_q is the limit q-Bernstein operator (1.4). Since $(q(1-x);\,q)_\infty$ is increasing on $[0,1]$, it follows that $\hat{g}_q$ is decreasing on $[0,1]$ and $\check{g}_q$ is increasing on $[0,1]$. At this stage, the monotonicity-preserving property of operators B_q is required. It follows immediately from [23, Theorem 7.5.8], which states that if f is an increasing (decreasing) function on $[0,1]$, the same is true for its q-Bernstein polynomials $B_{n,q}(f;\,x)$. Passing to limit as $n \to \infty$, one extends the monotonicity-preserving property on operators B_q. With this in mind, one concludes that $(D_q\hat{f}_a)(x)$ is decreasing on $[0,1]$. Likewise, $(D_q\check{f}_a)(x)$ is increasing on $[0,1]$. □

The following lemma puts the boundaries for the image D_qf, $f \in C[0,1]$.

Lemma 3.2. *Let $q \in (0,1)$, $f \in C[0,1]$, and $\|f\| \leqslant 1$. If $a = f(1)$, then*

$$(D_q\check{f}_a)(x) \leqslant (D_qf)(x) \leqslant (D_q\hat{f}_a)(x), \quad \text{for all } x \in [0,1].$$

Proof. The statement follows from the fact that

$$A_{k,q}(\check{f}_a) \leqslant A_{k,q}(f) \leqslant A_{k,q}(\hat{f}_a), \quad \text{for all } k \geqslant 0,$$

and the positivity of $p_k(q;\,x)$ on $[0,1]$. □

Corollary 3.3. *Let $q \in (0,1)$, $f \in C[0,1]$, and $\|f\| \leqslant 1$. If $a = f(1)$, then*

$$-1 + (1+a)(q;\,q)_\infty \leqslant (D_qf)(x) \leqslant 1 - (1-a)(q;\,q)_\infty.$$

The next statement provides an estimate for the distance between two limit q-Durrmeyer operators. Its sharpness will be established in Theorem 3.6.

Lemma 3.4. *Let $q,\ r \in [0,1]$ and $q > r$. Then, $\|D_q - D_r\| \leqslant 2 - 2(q;\,q)_\infty$.*

Proof. First, consider the case $q, r \in (0,1)$. By Corollary 3.3, one has, for all $x \in [0,1]$,

$$\lambda_1 \leqslant (D_qf)(x) \leqslant \lambda_2, \quad \mu_1 \leqslant (D_rf)(x) \leqslant \mu_2,$$

where

$$\lambda_1 = -1 + (1+a)(q;\,q)_\infty, \quad \lambda_2 = 1 - (1-a)(q;\,q)_\infty$$
$$\mu_1 = -1 + (1+a)(r;\,r)_\infty, \quad \mu_2 = 1 - (1-a)(r;\,r)_\infty.$$

The condition $q > r$ yields $(q;\,q)_\infty < (r;\,r)_\infty$, whence $[\mu_1, \mu_2] \subset [\lambda_1, \lambda_2]$. Therefore,

$$|(D_qf - D_rf)(x)| \leqslant \max\{\mu_2 - \lambda_1, \lambda_2 - \mu_1\}.$$

Now, for all $a \in [-1, 1]$, there holds:

$$\mu_2 - \lambda_1 = 2 - (q; q)_\infty - (r; r)_\infty + a[(r; r)_\infty - (q; q)_\infty] \leqslant 2 - 2(q; q)_\infty$$

and

$$\lambda_2 - \mu_1 = 2 - (q; q)_\infty - (r; r)_\infty - a[(r; r)_\infty - (q; q)_\infty] \leqslant 2 - 2(q; q)_\infty.$$

As a result, $\|D_q - D_r\| \leqslant 2 - 2(q; q)_\infty$, as claimed. For $r = 0$, one has

$$(D_q f)(x) - (D_0 f)(x) = (D_q f)(x) - a.$$

By virtue of Corollary 3.3, one writes

$$-2[1 - (q; q)_\infty] \leqslant -(1 + a)[1 - (q; q)_\infty] \leqslant (D_q f)(x) - (D_0 f)(x) \leqslant (1 - a)[1 - (q; q)_\infty] \leqslant 2[1 - (q; q)_\infty],$$

which leads to $\|D_q - D_0\| \leqslant 2 - 2(q; q)_\infty$. Finally, for $q = 1$, the desired estimate follows from the triangle inequality, as $\|D_q\| = 1$ for all $q \in [0, 1]$ (see (1.6)). □

Theorem 3.5. *The family of operators $\{D_q\}_{q\in[0,1]}$ is discontinuous with respect to q in the uniform operator topology at all $q \in (0, 1]$ and continuous at $q = 0$.*

Proof. (i) First, the discontinuity for arbitrary $r \in (0, 1)$ will be considered. Fix $r \in (0, 1)$ and select $q \in (r, 1)$ in a such a way that the time scales $\mathbb{J}_q$ and $\mathbb{J}_r$ defined by (3.2) obey the restriction $\mathbb{J}_q \cap \mathbb{J}_r = \{0, 1\}$. This holds whenever q is not a rational power of r. Next, for $M \in \mathbb{N}$, construct $f_M \in C[0, 1]$ satisfying the conditions:
- $\|f_M\| = 1$,
- $f_M(r^j) = -1$, for all $j \in \mathbb{N}_0$,
- $f_M(q^j) = \begin{cases} 1, & \text{for } 1 \leqslant j \leqslant M, \\ -1, & \text{for } j = 0 \text{ and } j \geqslant M + 1. \end{cases}$

This is possible since $\mathbb{J}_q \cap \mathbb{J}_r = \{0, 1\}$. Now, given $\varepsilon > 0$, opt for $M \in \mathbb{N}$ such that

$$S_{q,M,k} := \sum_{j=M+1}^{\infty} \frac{q^{j(k+1)}}{(q; q)_j} < \frac{\varepsilon}{2}.$$

Then,

$$\begin{aligned} A_{k,q}(f_M) &= (q^{k+1}; q)_\infty \sum_{j=0}^{\infty} \frac{f_M(q^j) q^{(k+1)j}}{(q; q)_j} \\ &= (q^{k+1}; q)_\infty \left[-1 + \sum_{j=1}^{M} \frac{q^{j(k+1)}}{(q; q)_j} - S_{q,M,k}\right] \\ &= (q^{k+1}; q)_\infty \left[-2 + \frac{1}{(q^{k+1}; q)_\infty} - 2S_{q,M,k}\right] \\ &= A_{k,q}(\hat{f}_{-1}) - 2(q^{k+1}; q)_\infty S_{q,M,k} =: A_{k,q}(\hat{f}_{-1}) + T_k, \end{aligned}$$

where $|T_k| = 2(q^{k+1}; q)_\infty S_{q,M,k} < \varepsilon$, $k \in \mathbb{N}_0$. Consequently,

$$(D_q f_M)(x) = (D_q \hat{f}_{-1})(x) + \sum_{k=0}^{\infty} T_k p_k(q; x).$$

Observe that $|\sum_{k=0}^{\infty} T_k p_k(q; x)| < \varepsilon$ for all $x \in [0, 1]$ due to Remark 1.4. Recall that $D_q \hat{f}_{-1}$ is a decreasing function on $[0, 1]$ with $(D_q \hat{f}_{-1})(0) = 1 - 2(q; q)_\infty$. Bearing in mind that $D_r f_M \equiv -1$, one arrives at

$$\|D_q - D_r\| \geqslant \|D_q f_M - D_r f_M\| \geqslant \|D_q \hat{f}_{-1} + 1\| - \varepsilon = 2 - 2(q; q)_\infty - \varepsilon.$$

As $\varepsilon > 0$ has been selected arbitrarily, one concludes, for every $q \in (r, 1)$ with $\mathbb{J}_q \cap \mathbb{J}_r = \{0, 1\}$, that

$$\|D_q - D_r\| \geqslant 2 - 2(q; q)_\infty. \tag{3.3}$$

To complete the proof, pick a sequence $\{q_n\} \to r^+$, so that $\{\mathbb{J}_{q_n} \cap \mathbb{J}_r\} = \{0, 1\}$ for all $n \in \mathbb{N}$. This can be fulfilled by setting $q_n = r^{\tau_n}$, where $\{\tau_n\} \to 1^-$ is a sequence of positive irrational numbers. Then, $\|D_{q_n} - D_r\| \geqslant 2 - 2(q_n; q_n)_\infty \to 2 - 2(r; r)_\infty \neq 0$ as $n \to \infty$. This reveals the discontinuity of $\{D_q\}_{q\in(0,1)}$ at every $q \in (0, 1)$.

(*ii*) To prove the discontinuity at $q = 1$, consider, for each $r \in (0, 1)$, the function $f_r \in C[0, 1]$ satisfying the conditions:
- $\|f_r\| = 1$,
- $f_r(r^j) = 1$, for all $j \in \mathbb{N}_0$,
- $f_r(r_0) = -1$, for some $r_0 \in (r, 1)$.

Then, $D_r f_r \equiv 1$, while $2 = \|D_r f_r - f_r\| \leqslant \|D_r - D_1\|$, implying that $\|D_r - D_1\| = 2$ for all $r \in (0, 1)$. Thence, $\{D_q\}_{q\in[0,1]}$ is discontinuous in the uniform operator topology whenever $q \in (0, 1]$.

(*iii*) To prove the continuity at $q = 0$, note that, by Lemma 3.4,

$$\|D_q - D_0\| \leqslant 2 - 2(q; q)_\infty \to 0, \quad \text{as } q \to 0^+.$$

This is because $\lim_{q\to0^+}(q; q)_\infty = 1$. The proof is complete. □

As a by-product of the previous reasoning, the next result on the distance between two q-Durrmeyer operators has been accomplished.

Theorem 3.6. *Let $q, r \in [0, 1]$ with $q > r$ and $\mathbb{J}_q \cap \mathbb{J}_r = \{0, 1\}$. Then,*

$$\|D_q - D_r\| = 2 - 2(q; q)_\infty.$$

Proof. The upper estimate for the distance between D_q and D_r is presented in Lemma 3.4. The lower estimate, when $r \neq 0$, has been obtained for $\mathbb{J}_q \cap \mathbb{J}_r = \{0, 1\}$ (see Formula (3.3) and the proof of Theorem 3.5 part (*ii*)). As for the case $r = 0$, applying Corollary 3.3 to $f = \hat{f}_{-1}$, one has

$$\|D_q - D_0\| \geqslant \|D_q\hat{f}_{-1} - D_0\hat{f}_{-1}\| = 2 - 2(q; q)_\infty.$$ □

4 Conclusion

This article deals with the one-parametric family $\{D_q\}_{q\in[0,1]}$ of Bernstein-type operators introduced by Gupta [12] and called by him the limit q-Durrmeyer operators. Although the approximation properties of those operators are examined with the help of commonly known tools accepted within the theory of positive linear operators, the presence of a parameter triggers an utterly innovative area of research related to the situations when the parameter varies. The current work falls exactly into this field. Going into details, the continuity of $\{D_q\}_{q\in[0,1]}$ with respect to q is investigated both in the strong and uniform operator topologies, which are the most important topologies in the operator theory. The main results of this study possess novelty and their proofs require new ideas. First, it is shown that the family of these operators is continuous in the strong operator topology for all $q \in [0, 1]$. Another important finding is that this family is continuous in the uniform operator topology only at $q = 0$; nevertheless, it is discontinuous at each $q \in (0, 1]$. In addition, the noteworthy fact that the sharp upper estimate for $\|D_q - D_r\|$ depends only on the maximum of q and r is established. Besides, the derived estimates imply that the equality $\|D_q - D_r\| = \|D_q\| + \|D_r\| = 2$ is attained if and only if one of the operators is D_0. These phenomena do not occur for the previously studied norms of the q-analogues [18,22]. The present piece of research paves the way to further investigate the norms-related properties of parametric families of Bernstein-type operators. We hope that our approach can be extended to other available versions of the q-Bernstein operators.

Acknowledgement: The authors express their sincere gratitude to the anonymous referees for their thorough reading of the manuscript and beneficial comments.

Funding information: The authors state that no funding involved.

Author contributions: All authors contributed equally to this work.

Conflict of interest: The authors state that there is no conflict of interest.

Data availability statement: Not applicable.

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