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# Disconnectedness of the subgraph $F^3$ for the group $\Gamma^3$

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Dedicated to Professor Hari M Srivastava

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## Abstract

In this paper we show that the subgraph  $F^3$  is disconnected and that for all integers  $m$ , we find all integers  $a$  and  $b$  such that  $(9m^2 - 4)a^2 + 4$  and  $5b^2 \pm 4$  are square. It turns out that the set of numbers  $b$  comprises the Fibonacci numbers.

**Keywords:** modular group; suborbital graph; disconnectedness; Fibonacci numbers

## 1 Introduction

Let  $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$  be the extended rationals, let  $\Gamma$  be the modular group acting on  $\hat{\mathbb{Q}}$  as with the upper half-plane  $\mathcal{H} = \{z \in \mathbb{C} : \text{Im } z > 0\}$ :

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z = \frac{x}{y} \rightarrow \frac{az + b}{cz + d} = \frac{ax + by}{cx + dy},$$

where  $a, b, c$ , and  $d$  are rational integers, and let  $\Gamma^3$  denote the group consisting of the cubes of the elements  $g$  of  $\Gamma$ , which is the group  $\{g \in \Gamma : ab + cd \equiv 0 \pmod{3}\}$ ; see [1].

Jones *et al.* [2] used the notion of the imprimitive action [3–5] for a  $\Gamma$ -invariant equivalence relation induced on  $\hat{\mathbb{Q}}$  by the congruence subgroup  $\Gamma_0(n) = \{g \in \Gamma : c \equiv 0 \pmod{n}\}$  to obtain some suborbital graphs and examined their connectedness and forest properties.

In [6], a  $\Gamma^3$ -invariant equivalence relation is introduced by using the subgroup  $\Gamma_0^3(n) = \{g \in \Gamma^3 : c \equiv 0 \pmod{n}\}$  to obtain suborbital graphs  $F_{u,n}^3$ . There, the connectivity properties of all subgraphs  $F_{u,n}^3$  other than  $F_{1,1}^3 = F^3$  are examined.

In this paper we show that the subgraph  $F^3$  is disconnected and give some results, which seem important from the point of view of number theory.

## 2 Preliminaries

Since  $\Gamma^3 = \{g^3 : g \in \Gamma\}$ , it is easily seen that the elements of  $\Gamma^3$  are ones of the forms  $\begin{pmatrix} 3a & b \\ c & 3d \end{pmatrix}$ ,  $\begin{pmatrix} a & 3b \\ 3c & d \end{pmatrix}$ , and  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  where  $a, b, c$ , and  $d \not\equiv 0 \pmod{3}$  in the third matrix. Furthermore,  $\Gamma_\infty^3 < \Gamma_0^3(n) \leq \Gamma^3$  for each positive integer  $n$ , where  $\Gamma_\infty^3$  is the stabilizer of  $\infty$  generated by the element  $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ , and second inclusion is strict if  $n > 1$ .

Since the group  $\Gamma^3$  is transitive on  $\hat{\mathbb{Q}}$  in [6], any reduced fraction  $\frac{r}{s}$  in  $\hat{\mathbb{Q}}$  equals  $g(\infty)$  for some  $g \in \Gamma^3$ . Similar to that of [2], we get the following  $\Gamma^3$ -invariant equivalence relation on  $\hat{\mathbb{Q}}$  by  $\Gamma_0^3(n)$  as  $\frac{r}{s} \sim \frac{x}{y}$  if and only if  $g^{-1}h \in \Gamma_0^3(n)$ , where  $g = \begin{pmatrix} r & * \\ s & * \end{pmatrix}$  and  $h$  is similar. Furthermore, the above equivalence relation is imprimitive, which means that it is different

from the identity relation ( $a \sim b$  if and only if  $a = b$ ) and the universal relation ( $a \sim b$  for all  $a, b \in \hat{\mathbb{Q}}$ ).

From the above defined  $\Gamma^3$ -invariant equivalence relation, we can verify that  $\frac{r}{s} \sim \frac{x}{y}$  if and only if  $ry - sx \equiv 0 \pmod{n}$ . The equivalence classes are called blocks, and a block containing the rational  $\frac{x}{y}$  is denoted by  $[\frac{x}{y}]$ .

Although the equivalence relations are resulting almost the same as in [2], the subgraph  $F_{1,1}$  in [2] is easily shown to be connected, but here we will see that the subgraph  $F_{1,1}^3$  is disconnected. So, using different subgroups changes the characters of the subgraphs.

### 3 Subgraphs $F_{u,n}^3$

The group  $\Gamma^3$  acts on  $\hat{\mathbb{Q}} \times \hat{\mathbb{Q}}$  through  $g : (\alpha, \beta) \rightarrow (g(\alpha), g(\beta))$ . The orbits are called sub-orbital. From the suborbital  $O^3(\alpha, \beta)$  containing  $(\alpha, \beta)$ , we can form the suborbital graph  $G^3(\alpha, \beta)$  whose vertices are the elements of  $\hat{\mathbb{Q}}$  and the edges are the pairs  $(a, b) \in O^3(\alpha, \beta)$ , which we denote by  $a \rightarrow b$  and represent as hyperbolic geodesics in  $\mathcal{H}$ .

Since  $\Gamma^3$  acts transitively on  $\hat{\mathbb{Q}}$ , every suborbital contains a pair  $(\infty, \frac{u}{n})$  for some  $\frac{u}{n}$  in  $\hat{\mathbb{Q}}$ ,  $n \geq 0$ ,  $(u, n) = 1$ . In this case, we denote the suborbital graph by  $G_{u,n}^3$  for short. From now on, we assume that  $n > 0$ .

As  $\Gamma^3$  permutes the blocks transitively, all subgraphs corresponding to the blocks are isomorphic, as in [2]. Therefore, we will only consider the subgraph  $F_{u,n}^3$  of  $G_{u,n}^3$  whose vertices form the block  $[\infty] = \{\frac{x}{y} \in \hat{\mathbb{Q}} \mid y \equiv 0 \pmod{n}\}$ . The following two results were proved in [6].

**Theorem 1**  $F_{u,n}^3 = F_{u',n'}^3$  if and only if  $n = n'$  and  $u \equiv u' \pmod{3n}$ .

**Theorem 2**  $\frac{r}{s} \rightarrow \frac{x}{y}$  is an edge in  $F^3 (= F_{1,1}^3)$  if and only if

1. if  $r \equiv 0 \pmod{3}$ , then  $y \equiv \pm s \pmod{3}$  and  $ry - sx = \pm 1$ , or
2. if  $s \equiv 0 \pmod{3}$ , then  $x \equiv \pm r \pmod{3}$ , and  $ry - sx = \pm 1$ , or
3. if  $r, s \not\equiv 0 \pmod{3}$ , then  $x \not\equiv \pm r \pmod{3}$ ,  $y \not\equiv \pm s \pmod{3}$  and  $ry - sx = \pm 1$ .

We can easily get the following lemmas.

**Lemma 1**  $\frac{r}{s} \rightarrow \frac{x}{y}$  is in  $F^3$  if and only if  $\frac{x}{y} \rightarrow \frac{r}{s}$  is in  $F^3$ .

**Lemma 2** [2] No edges of  $F^3$  cross in  $\mathcal{H}$ .

### 4 Disconnectedness of $F^3$

**Definition 1** For  $m \in \mathbb{N}$  and  $m \geq 2$ , let  $v_1, v_2, \dots, v_m$  be a finite sequence of vertices of  $F^3$ . Then the configuration  $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_m$  is called a finite path in  $F^3$ . A subgraph  $\wedge$  of  $F^3$  is called connected if every two vertices  $x$  and  $y$  of  $\wedge$  are connected by a finite path in  $F^3$ . Otherwise, we call  $\wedge$  disconnected.

Now we give one of our main theorems.

**Theorem 3** The graph  $F^3$  is disconnected.

We prove Theorem 3 after giving some theorems, propositions and lemmas as follows.

By Theorem 2, it is easily seen that the graph  $F^3$  is periodic with period 3. That is, if  $a \rightarrow b$  is in  $F^3$ , then  $a + 3m \rightarrow b + 3m$  is in  $F^3$  for all  $m \in \mathbb{Z}$ , and therefore, for some  $m$ ,

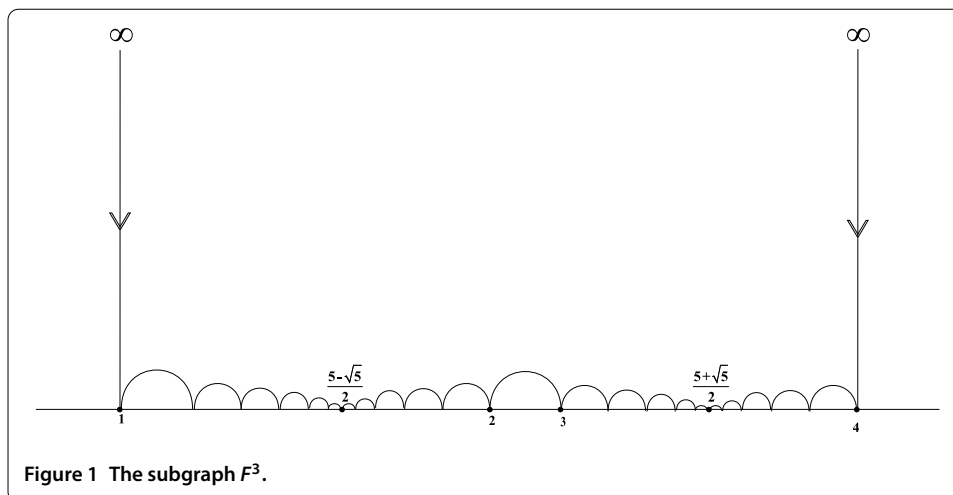


Figure 1 The subgraph  $F^3$ .

$a + m$  or  $b + m$  (not both) is  $\infty$  or both  $a + 3m$  and  $b + 3m$  are in the interval  $[1, 4]$ . Therefore we can only use the interval  $[1, 4]$  for our calculations as in Figure 1.

It is clear that  $T = \begin{pmatrix} 1 & -5 \\ 1 & -4 \end{pmatrix}$  is in  $\Gamma^3$  and the corresponding transformation  $T(z) = \frac{z-5}{z-4}$  is strictly increasing on  $[1, 4] \cap \mathbb{Q}$ . Furthermore, it is easily seen that  $T^m(\infty) \rightarrow T^m(\frac{1}{1})$  is an edge in  $F^3$  for all non-negative integers  $m$ . From this, we get, as an example, a finite path in  $F^3$  as  $\infty \rightarrow \frac{1}{1} \rightarrow \frac{4}{3} \rightarrow \frac{11}{8}$ .

**Lemma 3** *Let  $T$  be as above, then the sequence  $\{T^m(1)\}$  is strictly monotone increasing and  $T(\frac{1}{0}) \rightarrow T(\frac{1}{1}) \rightarrow T^2(\frac{1}{1}) \rightarrow \dots \rightarrow T^m(\frac{1}{1}) \rightarrow \dots$  is an infinite path in  $F^3$  in increasing order.*

*Proof* The conclusion follows from Theorem 2 and from the fact that  $T(z) = \frac{z-5}{z-4}$  is strictly increasing on  $[1, 4] \cap \mathbb{Q}$ .  $\square$

**Lemma 4** *Let  $a$  and  $b$  be in  $\mathbb{N}$  and let  $1 \leq \frac{a}{b} < \frac{5-\sqrt{5}}{2}$ , then  $\frac{a}{b} < T(\frac{a}{b}) < \frac{5-\sqrt{5}}{2}$ .*

*Proof* From  $\frac{a}{b} < \frac{5-\sqrt{5}}{2}$  we get  $2a - 5b < -\sqrt{5}b$ . Then squaring gives the inequality  $-a^2 + 4ab < -ab + 5b^2$ . That is,  $\frac{a}{b} < \frac{-a+5b}{-a+4b} = T(\frac{a}{b})$ . On the other hand,  $a^2 - 5ab + 5b^2 > 0$ , then it is easily seen that  $5(a - 4b)^2 < (3a - 10b)^2$ . As  $\frac{a}{b} < 2$ , then taking square roots gives  $\sqrt{5}(a - 4b) > 3a - 10b$ . This shows that  $T(\frac{a}{b}) < \frac{5-\sqrt{5}}{2}$ .  $\square$

**Proposition 1** *Let  $T$  be as above and  $1 \leq \frac{a}{b} < \frac{5-\sqrt{5}}{2}$ . Then  $\frac{a}{b} \rightarrow T(\frac{a}{b})$  is an edge in  $F^3$  if and only if there exists a natural number  $u$  such that  $u^2 = 5b^2 + 4$  and  $a = \frac{5b-\sqrt{5b^2+4}}{2}$ .*

*Proof* Let  $\frac{a}{b} \rightarrow T(\frac{a}{b})$  be an edge in  $F^3$ . Then, by using Theorem 2 and Lemma 4, we get  $a^2 - 5ab + 5b^2 - 1 = 0$ . Since  $\frac{a}{b} < \frac{5-\sqrt{5}}{2}$ , we have  $a = \frac{5b-\sqrt{5b^2+4}}{2}$ . This concludes that  $\sqrt{5b^2 + 4}$  is an integer  $u$ .

Conversely, it is clear to see  $M = \begin{pmatrix} -5b+\sqrt{5b^2+4} & 5b \\ -b & 5b+\sqrt{5b^2+4} \end{pmatrix}$  is in  $\Gamma^3$  and that  $M(\frac{1}{0}) = \frac{a}{b}$  and  $M(\frac{1}{1}) = T(\frac{a}{b})$ . Therefore, by the definition of edges of  $F^3$ , the configuration  $\frac{a}{b} \rightarrow T(\frac{a}{b})$  is an edge in  $F^3$ .  $\square$

**Theorem 4** *The positive rational number  $\frac{x}{y}$  is in  $\{T^m(\frac{1}{0}) : m \in \mathbb{N}\} = A$  if and only if there exists a natural number  $u$  such that  $5y^2 + 4 = u^2$  and  $x = \frac{5y-\sqrt{5y^2+4}}{2}$ .*

*Proof* From Proposition 1, the ‘if’ part is clear.

Conversely, we show that under the hypothesis,  $\frac{x}{y}$  is in the set  $A$ . Since  $T^m(\frac{1}{0}) = \frac{5y - \sqrt{5y^2 + 4}}{y}$  for any  $m \in \mathbb{N}$ ,  $\lim_{m \rightarrow \infty} T^m(\frac{1}{0}) = \frac{5 - \sqrt{5}}{2}$ . As  $1 \leq \frac{x}{y} < \frac{5 - \sqrt{5}}{2}$ , if  $\frac{x}{y}$  is not in  $A$ , there exists  $k \in \mathbb{N}$  such that  $T^k(\frac{1}{0}) < \frac{x}{y} < T^{k+1}(\frac{1}{0})$ . From this, we get

$$T^k\left(\frac{1}{0}\right) = \frac{5a - \sqrt{5a^2 + 4}}{2} < \frac{5y - \sqrt{5y^2 + 4}}{2} < 1 + \frac{a}{3a - \frac{3a - \sqrt{5a^2 + 4}}{2}} = \frac{5a - v}{2} < \frac{5y - u}{2} < \frac{5a + v}{2}.$$

We know that  $\frac{5y - u}{2} \rightarrow T(\frac{5y - u}{y}) = 1 + \frac{y}{3y - \frac{5y - u}{2}}$  is an edge in  $F^3$ . From Lemma 2 it must be that  $T(\frac{5y - u}{y}) = \frac{5y + u}{2} < \frac{5a + v}{2}$ . This gives the inequality  $vy < au$ . But, from  $\frac{5a - v}{2} < \frac{5y - u}{2}$ , we arrive at the inequality  $vy > au$ , a contradiction. This concludes the proof of the theorem.  $\square$

**Corollary 1**  $\frac{1}{0} \rightarrow 0 + \frac{1}{1} \rightarrow 1 + \frac{1}{3} \rightarrow 1 + \frac{3}{8} \xrightarrow{T} \dots \xrightarrow{T} 1 + \frac{a_n}{b_n} \xrightarrow{T} 1 + \frac{b_n}{3b_n - a_n} \xrightarrow{T} \dots$  is an infinite path in  $F^3$ , and all vertices of the path are smaller than  $\frac{5 - \sqrt{5}}{2}$ , and the natural numbers  $x$  and  $y$  in the vertex  $1 + \frac{x}{y}$  are such that  $5x^2 + 4$  and  $5y^2 + 4$  are square. Furthermore,  $a_n = \frac{3b_n - \sqrt{5b_n^2 + 4}}{2}$  and  $\lim_{n \rightarrow \infty} (1 + \frac{a_n}{b_n}) = \frac{5 - \sqrt{5}}{2}$ , where  $T$  is as above.

*Proof* Lemma 3 and Theorem 4 conclude the proof.  $\square$

If we follow the way of the above, we arrive at the following two results without proofs.

**Theorem 5** Let  $T = \begin{pmatrix} -1 & 5 \\ -1 & 4 \end{pmatrix}$  and  $\frac{5 - \sqrt{5}}{2} < \frac{a}{b} \leq 2$ . Then  $\frac{5 - \sqrt{5}}{2} < T(\frac{a}{b}) < \frac{a}{b}$  and  $\frac{a}{b} \rightarrow T(\frac{a}{b})$  is an edge in  $F^3$  if and only if  $5b^2 - 4$  is a square and  $a = \frac{5b - \sqrt{5b^2 - 4}}{2}$ .

**Corollary 2**  $3 \xrightarrow{T} 2 = 1 + \frac{1}{1} \xrightarrow{T} 1 + \frac{1}{2} \xrightarrow{T} 1 + \frac{2}{5} \xrightarrow{T} \dots \xrightarrow{T} 1 + \frac{a_n}{b_n} \xrightarrow{T} 1 + \frac{b_n}{3b_n - a_n} \xrightarrow{T} \dots$  is an infinite path in  $F^3$  in decreasing order such that all vertices of the path are greater than  $\frac{5 - \sqrt{5}}{2}$ , and the natural numbers  $x$  and  $y$  in the vertex  $1 + \frac{x}{y}$  are such that  $5x^2 - 4$  and  $5y^2 - 4$  are squares. Furthermore,  $a_n = \frac{3b_n - \sqrt{5b_n^2 - 4}}{2}$  and  $\lim_{n \rightarrow \infty} (1 + \frac{a_n}{b_n}) = \frac{5 - \sqrt{5}}{2}$ , where  $T$  is as above.

**Theorem 6** Let  $k$  be a natural number and let the vertices  $v_1, v_2, \dots, v_k$  in  $[1, 3]$  of  $F^3$  be such that at least one is smaller and one is greater than  $\frac{5 - \sqrt{5}}{2}$ . Then the path  $\infty \rightarrow v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k$  does not occur in  $F^3$ .

*Proof* If the above situation occurs, since  $\frac{5 - \sqrt{5}}{2}$  is not a vertex in  $F^3$ , there exists  $1 \leq m < k$  such that  $v_m < \frac{5 - \sqrt{5}}{2} < v_{m+1}$  and  $v_m \rightarrow v_{m+1}$  is an edge in  $F^3$ .

Since the sequences of the vertices in Corollary 1 and Corollary 2 converge to  $\frac{5 - \sqrt{5}}{2}$ , Lemma 2 gives that there exist naturals  $m$  and  $n$  such that

$$T^m\left(\frac{1}{0}\right) = v_m \rightarrow v_{m+1} = T^n\left(\frac{3}{1}\right).$$

Suppose first that  $n > m$ . Then, multiplying by  $T^{-m}$  and using Theorem 5, we have  $\frac{1}{0} \rightarrow T^{n-m}(\frac{3}{1}) = \frac{5b - \sqrt{5b^2 - 4}}{b}$  for some  $b \in \mathbb{N}$ . Therefore, from Theorem 2, we get  $b = 1$ . That is,  $\frac{1}{0} \rightarrow \frac{3}{1}$  is an edge in  $F^3$ , a contradiction.

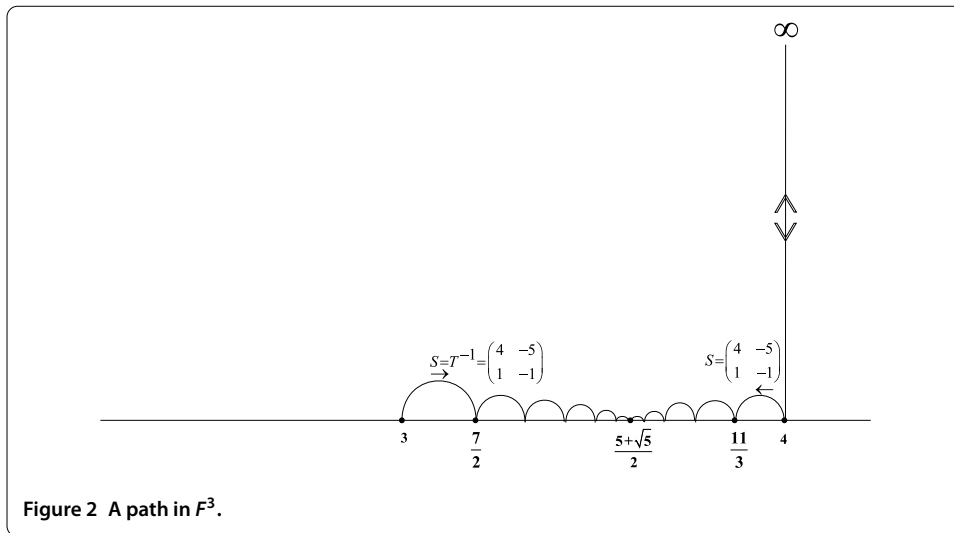


Figure 2 A path in  $F^3$ .

Suppose now that  $m > n$ . Then, again multiplying by  $T^{-n}$ , we have  $T^{m-n}(\frac{1}{0}) \rightarrow \frac{3}{1}$ . In this case,  $T^{m-n}(\frac{1}{0}) \geq 2$ . But, in any case,  $T^{m-n}(\frac{1}{0}) < \frac{5-\sqrt{5}}{2}$ , a contradiction.

Finally, let  $m = n$ . Then  $\frac{1}{0} \rightarrow \frac{3}{1}$  must be an edge in  $F^3$ , a contradiction. These complete the proof.  $\square$

Let  $S = T^{-1} = \begin{pmatrix} 4 & -5 \\ 1 & -1 \end{pmatrix}$ . Then likewise we do before we give the following five results without a proof.

**Lemma 5** Let  $3 \leq \frac{a}{b} < \frac{5+\sqrt{5}}{2}$ . Then  $\frac{a}{b} < S(\frac{a}{b}) < \frac{5+\sqrt{5}}{2}$ . Furthermore,  $\frac{a}{b} \rightarrow S(\frac{a}{b})$  is an edge in  $F^3$  if and only if  $\sqrt{5b^2 - 4}$  is a natural number and  $a = \frac{5b + \sqrt{5b^2 - 4}}{2}$ .

**Corollary 3**  $3 = 4 - \frac{1}{1} \xrightarrow{S} 4 - \frac{1}{2} \xrightarrow{S} 4 - \frac{2}{5} \xrightarrow{S} \dots \xrightarrow{S} 4 - \frac{3b - \sqrt{5b^2 - 4}}{b} \xrightarrow{S} \dots$  is an infinite path in  $F^3$  in increasing order, as seen Figure 2, and the limit of the sequence of vertices is  $\frac{5+\sqrt{5}}{2}$ .

**Lemma 6** Let  $\frac{5+\sqrt{5}}{2} < \frac{a}{b} \leq 4$ . Then  $\frac{5+\sqrt{5}}{2} < S(\frac{a}{b}) < \frac{a}{b}$ . Furthermore,  $\frac{a}{b} \rightarrow S(\frac{a}{b})$  is an edge in  $F^3$  if and only if  $5b^2 + 4$  is a square and  $a = \frac{5b + \sqrt{5b^2 + 4}}{2}$ .

**Lemma 7**  $4 - \frac{0}{1} \xrightarrow{S} 4 - \frac{1}{3} \xrightarrow{S} 4 - \frac{3}{8} \xrightarrow{S} \dots \xrightarrow{S} 4 - \frac{3b - \sqrt{5b^2 + 4}}{b} \xrightarrow{S} \dots$  is an infinite path in  $F^3$  in decreasing order, and the limit of the sequence of vertices is  $\frac{5+\sqrt{5}}{2}$ .

**Theorem 7** Let  $k$  be a natural number and let the vertices  $v_1, v_2, \dots, v_k$ , in  $[3, 4]$ , of  $F^3$  be such that at least one is smaller and one is greater than  $\frac{5+\sqrt{5}}{2}$ . Then  $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k \rightarrow \infty$  does not occur in  $F^3$ .

*Proof* We conclude the proof as in Theorem 6.  $\square$

*Proof of Theorem 3* Theorems 6 and 7 conclude that the vertices of  $F^3$  in  $(\frac{5-\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2})$  are not connected to the vertex  $\infty$ . That is, the graph  $F^3$  is disconnected.  $\square$

We finally give one of our main results as follows.

**Theorem 8** For all natural numbers  $m$ , the natural numbers  $b$  that make the number  $(9m^2 - 4)b^2 + 4$  square are  $0, 1, 3m, 9m^2 - 1, 3m(9m^2 - 1) - 3m, \dots, a, b, 3mb - a, \dots$

*Proof* For the proof, we only use, as above, the interval  $[1, 4]$ . It is clear that the matrix  $M = \begin{pmatrix} -1 & 3m+2 \\ -1 & 3m+1 \end{pmatrix}$  is in  $F^3$ . Theorem 2 gives that  $\frac{1}{1} \rightarrow M(\frac{1}{1}) = \frac{3m+1}{3m}$  is an edge in  $F^3$ . Since the transformation  $M(x) = \frac{-x+3m+2}{-x+3m+1}$  is an increasing function on  $[1, 4]$  and  $\frac{1}{1} < M(\frac{1}{1}) = \frac{3m+1}{3m}$ , we can easily see that, for all  $k \in \mathbb{N}$ ,  $M^k(\frac{1}{1}) < M^{k+1}(\frac{1}{1})$ . That is, the sequence  $\{M^k(1)\}$  is an increasing sequence. Furthermore,  $M^k(1) = [1; \underbrace{3m, 3m, 3m, \dots, 3m}_k]$ , or

$$M^k(1) = 1 + \frac{1}{3m - \frac{1}{3m - \frac{1}{3m - \dots - \frac{1}{3m}}}}$$

And more,  $\lim_{k \rightarrow \infty} M^k(1) = \frac{3m+2-\sqrt{9m^2-4}}{2}$ . So, for all  $k \in \mathbb{N}$ ,  $M^k(1) < \frac{3m+2-\sqrt{9m^2-4}}{2}$ . If  $\frac{a}{b} < \frac{3m+2-\sqrt{9m^2-4}}{2}$  we can easily see that  $T(\frac{a}{b}) < \frac{3m+2-\sqrt{9m^2-4}}{2}$ , and furthermore, if  $\frac{a}{b} \rightarrow T(\frac{a}{b}) = \frac{-a+(3m+2)b}{-a+(3m+1)b}$  is an edge in  $F^3$ , then, by Theorem 2,  $a^2 - (3m+2)b + (3m+2)b^2 - 1 = 0$ . Solving the equation, we have  $a = \frac{(3m+2)b - \sqrt{(9m^2-4)b^2+4}}{2}$ , where we get the sign ‘-’ since for all  $k \in \mathbb{N}$ ,  $M^k(1) < \frac{3m+2-\sqrt{9m^2-4}}{2}$ . Because  $a$  is an integer,  $\sqrt{(9m^2-4)b^2+4}$  must be an integer. According to Theorem 2, for all  $k \in \mathbb{N}$ ,  $M^k(\frac{1}{1}) \rightarrow M^{k+1}(\frac{1}{1})$  is an edge in  $F^3$ . Therefore

$$\frac{1}{1} \xrightarrow{M} M\left(\frac{1}{1}\right) \xrightarrow{M} M^2\left(\frac{1}{1}\right) \xrightarrow{M} \dots \xrightarrow{M} \left(\frac{a}{b}\right) \xrightarrow{M} M\left(\frac{a}{b}\right) \xrightarrow{M} \dots$$

is an infinite path  $\gamma$  in  $F^3$ . All denominators of vertices  $\frac{a}{b}$  of  $\gamma$  make  $(9m^2 - 4)b^2 + 4$  square. We can rewrite  $\gamma$  as

$$1 + \frac{0}{1} \rightarrow 1 + \frac{1}{3m} \rightarrow 1 + \frac{3m}{9m^2 - 1} \rightarrow 1 + \frac{9m^2 - 1}{3m(9m^2 - 1) - 3m} \rightarrow \dots \rightarrow 1 + \frac{a'}{b'} \rightarrow 1 + \frac{b'}{3mb' - a'} \rightarrow \dots$$

And we conclude that the numbers  $0, 1, 3m, 9m^2 - 1, 3m(9m^2 - 1) - 3m, \dots, a_0, b_0, 3mb_0 - a_0, \dots$  make  $(9m^2 - 4)b^2 + 4$  square.

Let us now show the only non-negative integers  $b$  such that  $(9m^2 - 4)b^2 + 4$  is square.

Conversely, suppose that there is a natural number  $t$  such that  $(9m^2 - 4)t^2 + 4$  is square.

Then  $\frac{a_1}{b_1} = \frac{(3m+2)t - \sqrt{(9m^2-4)t^2+4}}{t}$  is smaller than  $\frac{3m+2-\sqrt{9m^2-4}}{2}$  and, from Theorem 2, we get that  $\frac{a_1}{b_1} \rightarrow T(\frac{a_1}{b_1})$  is an edge in  $F^3$ . Suppose, for some  $k \in \mathbb{N}$ ,  $M^k(\frac{1}{1}) < \frac{a_1}{b_1} < M^{k+1}(\frac{1}{1})$ . Lemma 2 says that  $T^m(\frac{1}{1}) < \frac{a_1}{b_1} < T^{m+1}(\frac{a_1}{b_1}) < M^{k+1}(\frac{1}{1})$ . Therefore, for some  $y$ ,

$$\begin{aligned} M^k\left(\frac{1}{1}\right) &= 1 + \frac{3my - \sqrt{(9m^2-4)y^2+4}}{2} < \frac{a_1}{b_1} = 1 + \frac{3mt - \sqrt{(9m^2-4)t^2+4}}{2} \\ &\rightarrow T\left(\frac{a_1}{b_1}\right) = 1 + \frac{t}{3mt - \frac{3mt - \sqrt{(9m^2-4)t^2+4}}{2}} < M^{k+1}\left(\frac{1}{1}\right) \\ &= 1 + \frac{y}{3my - \frac{3my - \sqrt{(9m^2-4)y^2+4}}{2}} \end{aligned}$$

From the first inequality, we conclude that  $t\sqrt{(9m^2-4)y^2+4} > y\sqrt{(9m^2-4)y^2+4}$ . From the second inequality, we just have  $t\sqrt{(9m^2-4)y^2+4} < y\sqrt{(9m^2-4)y^2+4}$ , a contradiction. Consequently,  $\frac{a_1}{b_1}$  must be in the set  $\{M^k(\frac{1}{1}) : k \in \mathbb{N}\}$ . This completes the proof of the theorem.  $\square$

From Corollaries 1 and 2, we get the following without a proof.

**Corollary 4** *The non-negative integers  $b$  such that  $5b^2 + 4$  is square are  $0, 1, 3, 8, 21, 55, 144, \dots, a, b, 3b - a, \dots$ .*

**Corollary 5** *The non-negative integers  $b$  making  $5b^2 - 4$  square are  $1, 2, 5, 13, 34, 89, \dots, a, b, 3b - a, \dots$ .*

From Corollaries 4 and 5, we conclude the following important corollary.

**Corollary 6** *Let  $\{a_n\}$  and  $\{b_n\}$  be the sequences  $(0, 1, 3, \dots, a, b, 3b - a, \dots)$  and  $(1, 2, 5, \dots, c, d, 3d - c, \dots)$ , respectively. Then the sequence  $(a_1, b_1, a_2, b_2, \dots, a_n, b_n, \dots)$  is the Fibonacci sequence.*

*Proof* Let us see that  $a_n + b_n = a_{n+1}$  and  $b_n + a_{n+1} = b_{n+1}$  for all  $n$  in  $\mathbb{N}$  by induction. Suppose that the assertion is true up to the natural number  $k$ . Let us show that  $a_{k+1} + b_{k+1} = a_{k+2}$ . Since  $a_{k+1} = 3a_k - a_{k-1}$  and  $b_{k+1} = 3b_k - b_{k-1}$ ,  $a_{k+1} + b_{k+1} = 3(a_k + b_k) - (a_{k-1} + b_{k-1}) = 3a_{k+1} - a_k = a_{k+2}$ . The other is similar.  $\square$

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors completed the paper together, read and approved the final manuscript.

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