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Conditions to be a Forest for Normalizer

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Abstract

In this paper, we examine some suborbital graphs for the normalizer of $\Gamma_0(N)$ in $PSL(2,\mathbb{R})$.

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1. Introduction

Let PSL(2,$\mathbb{R}$) denote the group of all linear fractional transformations

$$T : z \to \frac{az+b}{cz+d}\ \text{, where } a, b, c \text{ and } d \text{ are real and } ad - bc = 1.$$

In terms of matrix representation, the elements of PSL(2,$\mathbb{R}$) correspond to the matrices

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1.$$

This is the automorphism group of the upper half plane $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im(z)} > 0\}$. Γ, the modular group, is the subgroup of PSL(2,$\mathbb{R}$) such that a, b, c and d are integers. $\Gamma_0(N)$ is the subgroup of Γ with $N|c$.

In [1], the normalizer $Nor(N)$ of $\Gamma_0(N)$ in PSL(2,$\mathbb{R}$) consists exactly of matrices

$$\begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix},$$

where $e \parallel \frac{N}{h^2}$ and h is the largest divisor of 24 for which $h^2|N$ with understandings

that the determinant e of the matrix is positive, and that $r \parallel s$ means that $r|s$ and $(r, s/r) = 1$ (r is called an exact divisor of s). $Nor(N)$ is a Fuchsian group whose fundamental domain has finite area, so it has a signature consisting of the geometric invariants

$$(g; m_1, ..., m_r, s)$$

where g is the genus of the compactified quotient space, $m_1, ..., m_r$ are the periods of the elliptic elements and s is the parabolic class number.

2 The Action of $Nor(N)$ on $\hat{\mathbb{Q}}$

Every element of the extended set of rationals $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ can be represented as a reduced fraction $\frac{x}{y}$, with $x, y \in \mathbb{Z}$ and $(x, y) = 1$. ∞ is represented as $\frac{1}{0} = \frac{-1}{0}$. The action of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ on $\frac{x}{y}$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \frac{x}{y} \to \frac{ax+by}{cx+dy}.$$

Lemma 2.1 ([1]) *Let N have the prime power decomposition as $2^{\alpha_1} \cdot 3^{\alpha_2} \cdot p_3^{\alpha_3} \cdots p_r^{\alpha_r}$. Then $Nor(N)$ acts transitively on $\hat{\mathbb{Q}}$ if and only if $\alpha_1 \leq 7$, $\alpha_2 \leq 3$ and $\alpha_i \leq 1$ for $i = 3, \ldots, r$.* ∎

In this study, N will be of the form $2^\alpha p^2$, where $\alpha \geq 8$ and p is prime > 3. Clearly, $Nor(2^\alpha p^2)$ is not transitive on $\hat{\mathbb{Q}}$. Therefore, we will find a maximal subset of $\hat{\mathbb{Q}}$ on which $Nor(2^\alpha p^2)$ acts transitively. For this, we give some well-known facts as following lemmas about the orbits of the action of $\Gamma_0(N)$ on $\hat{\mathbb{Q}}$ without proofs.

Lemma 2.2 *Let k/s be an arbitrary rational number with $(k, s) = 1$. Then there exists some element $A \in \Gamma_0(N)$ such that $A(k, s) = (k_1, s_1)$ with $s_1|N$ transitive.*

Lemma 2.3 *Let $d|N$. Then the orbit $\begin{pmatrix} a \\ d \end{pmatrix}$ of a/d with $(a, d) = 1$ under $\Gamma_0(N)$ is the set $\left\{x/y \in \hat{\mathbb{Q}} : (N, y) = d, a \equiv x\frac{y}{d} mod(d, N/d)\right\}$. Furthermore the number of orbits $\begin{pmatrix} a \\ d \end{pmatrix}$ with $d|N$ under $\Gamma_0(N)$ is just $\varphi(d, N/d)$ where φ is Euler's functions.*

Corollary 2.4 *Let $d|N$ and let $(a_1, d) = (a_2, d) = 1$. Then $\begin{pmatrix} a_1 \\ d \end{pmatrix}$ and $\begin{pmatrix} a_2 \\ d \end{pmatrix}$ are conjugate under $\Gamma_0(N)$ iff $a_1 = a_2 \ mod(d, N/d)$.*

If one can just examine the actions of the elements of $Nor(2^\alpha p^2)$ on the orbit $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, the following result is easily obtained:

Theorem 2.5 *The set $\hat{\mathbb{Q}}(2^\alpha p^2) := \bigcup \begin{pmatrix} a \\ b \end{pmatrix}$, where $\begin{pmatrix} a \\ b \end{pmatrix}$ is as in Lemma2.3; $b = 2^i p^j$ or $2^{\alpha - i} p^j$; $i = 0, 1, 2, 3$ and $j = 0, 2$, is an orbit of $Nor(2^\alpha p^2)$ on $\hat{\mathbb{Q}}$.*

Therefore the set $\hat{\mathbb{Q}}(2^\alpha p^2)$ is one on which $Nor(2^\alpha p^2)$ acts transitively. We now consider the imprimitivity of the action of $Nor(2^\alpha p^2)$ on $\hat{\mathbb{Q}}(2^\alpha p^2)$, beginning with a general discussion of primitivity of permutation groups. Let (G, Δ) be a transitive permutation group, consisting of a group G acting on a set Δ transitively. An equivalence relation $\approx$ on Δ is called *G-invariant* if, whenever $\alpha, \beta \in \Delta$ satisfy $\alpha \approx \beta$, then $g(\alpha) \approx g(\beta)$ for all $g \in G$. The equivalence classes are called blocks, and the block containing α is denoted by $[\alpha]$.

We call (G, Δ) *imprimitive* if Δ admits some G-invariant equivalence relation different from

(i) the identity relation, $\alpha \approx \beta$ if and only if $\alpha = \beta$;
(ii) the universal relation, $\alpha \approx \beta$ for all $\alpha, \beta \in \Delta$.

Otherwise (G, Δ) is called *primitive*. These two relations are supposed to be trivial relations. Clearly, a primitive group must be transitive, for if not the orbits would form a system of blocks. The converse is false, but we have the following useful result.

Lemma 2.6 ([3]). *Let (G, Δ) be a transitive permutation group. (G, Δ) is primitive if and only if G_α,the stabilizer of $\alpha \in \Delta$, is a maximal subgroup of G for each $\alpha \in \Delta$.*

From the above lemma we see that whenever, for some α, $G_\alpha \lneqq H \lneqq G$, then Ω admits some G-invariant equivalence relation other than the trivial cases. Because of the transitivity, every element of Ω has the form $g(\alpha)$ for some $g \in G$. Thus one of the non-trivial G-invariant equivalence relation on Ω is given as follows:

$$g(\alpha) \approx g'(\alpha) \text{ if and only if } g' \in gH.$$

The number of blocks (equivalence classes) is the index $|G : H|$ and the block containing α is just the orbit $H(\alpha)$.

We can apply these ideas to the case where G is the $Nor(2^\alpha p^2)$ and Δ is $\hat{\mathbb{Q}}(2^\alpha p^2)$ which is the orbit in Theorem 1, G_α is the stabilizer of ∞ in $\hat{\mathbb{Q}}(2^\alpha p^2)$; that is, $G_\infty = \left\langle \begin{pmatrix} 1 & 1/2^3 \\ 0 & 1 \end{pmatrix} \right\rangle$, and H is $N_0 = \left\langle \Gamma_0(2^\alpha p^2), \begin{pmatrix} a & b/2^3 \\ 2^{\alpha-3}p^2 c & d \end{pmatrix}, \begin{pmatrix} 2^{\alpha-6}a & b/2^3 \\ 2^{\alpha-3}p^2 c & 2^{\alpha-6}d \end{pmatrix} \right\rangle$.

Clearly $G_\infty < N_0 < Nor(2^\alpha p^2)$.

Lemma 2.7 ([1]) *The index $|Nor(N) : \Gamma_0(N)| = 2^\rho h^2 \tau$, where ρ is the number of prime factors of N/h^2, $\tau = (\frac{3}{2})^{\varepsilon_1}(\frac{4}{3})^{\varepsilon_2}$,*

$$\varepsilon_1 = \begin{cases} 1 & \text{if } 2^2, 2^4, 2^6 \parallel N \\ 0 & \text{otherwise} \end{cases}, \quad \varepsilon_2 = \begin{cases} 1 & \text{if } 9 \parallel N \\ 0 & \text{otherwise} \end{cases} \blacksquare$$

Using the Lemma 2.7, we get following easily:

Theorem 2.8 *There are only two blocks which are $[\infty]$ and $[0]$. The first(or second) is the subset of $\hat{\mathbb{Q}}(2^\alpha p^2)$ where $j = 2$(or $j = 0$) in Theorem 2.5.*

3 Suborbital Graphs of $Nor(2^\alpha p^2)$ on $\hat{\mathbb{Q}}(2^\alpha p^2)$

In[6], Sims introduced the idea of the suborbital graphs of a permutation group G acting on a set Δ , these are graphs with vertex-set Δ, on which G induces automorphisms. We summarise Sims'theory as follows: Let (G, Δ) be transitive permutation group. Then G acts on $\Delta \times \Delta$ by $g(\alpha, \beta) = (g(\alpha), g(\beta))(g \in$

$G, \alpha, \beta \in \Delta$). The orbits of this action are called *suborbitals* of G. The orbit containing (α, β) is denoted by $O(\alpha, \beta)$. From $O(\alpha, \beta)$ we can form a *suborbital graph* $G(\alpha, \beta)$: its vertices are the elements of Δ, and there is a directed edge from γ to δ if $(\gamma, \delta) \in O(\alpha, \beta)$. A directed edge from γ to δ is denoted by $(\gamma \to \delta)$. If $(\gamma, \delta) \in O(\alpha, \beta)$, then we will say that there exists an edge $(\gamma \to \delta)$ in $G(\alpha, \beta)$.

If $\alpha = \beta$, the corresponding suborbital graph $G(\alpha, \alpha)$, called the trivial suborbital graph, is *self-paired*: it consists of a loop based at each vertex $\alpha \in \Delta$. By a *circuit* of length m (or an closed edge path), we mean a sequance $\nu_1 \to \nu_2 \to \cdots \to \nu_m \to \nu_1$ such that $\nu_i \neq \nu_j$ for $i \neq j$, where $m \geq 3$. If $m = 3$ or 4 then the circuit is called a triangle or rectangle. We call a graph a *forest* if it does not contain any circuits.

In this study, G and Δ will be $Nor(N)$ and $\hat{\mathbb{Q}}$, respectively. All circuits in suborbital graph for $Nor(N)$ where N is a squre-free positive integer was studied in [4,5]. We now investigate the suborbital graphs for the action $Nor(2^\alpha p^2)$ on $\hat{\mathbb{Q}}(2^\alpha p^2)$. Since the action $Nor(2^\alpha p^2)$ on $\hat{\mathbb{Q}}(2^\alpha p^2)$ is transitive, $Nor(2^\alpha p^2)$ permutes the blocks transitively; so the subgraphs are all isomorphic. Hence it is sufficent to study with only one block. On the other hand, it is clear that each non-trivial suborbital graph contains a pair $(\infty, u/2^\alpha p^2)$ for some $u/2^\alpha p^2 \in \hat{\mathbb{Q}}(2p^2)$. Therefore, we work on the following case: We denote by $F(\infty, u/2^\alpha p^2)$ the subgraph of $G(\infty, u/2^\alpha p^2)$ such that its vertical are in the block $[\infty]$.

Theorem 3.1 *Let r/s and x/y be in the block $[\infty]$. Then there is an edge $r/s \to x/y$ in $F(\infty, u/2^\alpha p^2)$ iff*

(i) *If $2^{\alpha-k}p^2 \parallel s$, then $x \equiv \pm ur (mod 2^{\alpha-3}p^2), y \equiv \pm us (mod 2^\alpha p^2), ry - sx = \pm 2^\alpha p^2$,*

(ii) *If $2^k p^2 \parallel s$, then $x \equiv \pm 2^{3-k} ur (mod 2^k p^2), y \equiv \pm 2^{3-k} us (mod 2^\alpha p^2), ry - sx = \pm 2^{2\alpha - 6} p^2$, where $0 \leq k \leq 3$ and $k \in \mathbb{Z}$.*

Proof. We prove first (i).Assume first that $r/s \to x/y$ is an edge in $F(\infty, u/p^2)$, $0 \leq k \leq 3$ and $k \in \mathbb{Z}$. It means that there exists some T in the normalizer $Nor(2^\alpha p^2)$ such that T sends the pair $(\infty, u/2^\alpha p^2)$ to the pair $(r/s, x/y)$, that is $T(\infty) = r/s$ and $T(u/2^\alpha p^2) = x/y$. Since $2^{\alpha-k}p^2 \parallel s$, T must be of the form A_1 where a and d are odd. $T(\infty) = \dfrac{a}{2^{\alpha-3}p^2 c} = \dfrac{r}{s}$ gives that $r = a$ and $s = 2^{\alpha-3}p^2 c$. $T(u/2^\alpha p^2) = \dfrac{au + 2^{\alpha-3}bp^2}{2^{\alpha-3}p^2 cu + 2^\alpha dp^2} = \dfrac{r}{s}$ gives that $x \equiv \pm ur (mod 2^{\alpha-3}p^2), y \equiv \pm us (mod 2^\alpha p^2)$. Furthermore, we get $ry - sx = \pm 2^\alpha p^2$ from the equation

$$\begin{pmatrix} a & b/2^3 \\ 2^{\alpha-3}p^2 c & d \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 2^\alpha p^2 \end{pmatrix} = \begin{pmatrix} r & s \\ x & y \end{pmatrix} .$$

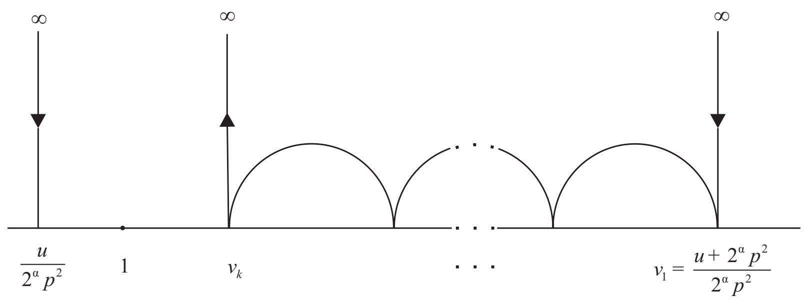

Figure 1: Path of the action

For the opposite direction, we assume that $2^{\alpha-k}p^2 \parallel s$ where $0 \leq k \leq 3$ and $k \in \mathbb{Z}$, and $x \equiv \pm ur(mod 2^{\alpha-3}p^2), y \equiv \pm us(mod 2^{\alpha}p^2), ry - sx = \pm 2^{\alpha}p^2$. In this case, there exist $b, d \in \mathbb{Z}$ such that $x = ur + 2^{\alpha-3}bp^2$ and $y = us + 2^{\alpha}dp^2$. If we put these equivalences in $ry - sx = 2^{\alpha}p^2$, we obtain $rd - (b/2^3)s = 1$. So the element $T_0 = \begin{pmatrix} r & b/2^3 \\ s & d \end{pmatrix}$ is clearly in N_0. For (ii), taking the element of the form $\begin{pmatrix} 2^{\alpha-6}a & b/2^3 \\ 2^{\alpha-3}p^2c & 2^{\alpha-6}d \end{pmatrix}$ where $a = 2^{\alpha-3}a_0$, and a_0, b, c are odd, similiar calculations are done. ∎

Now, let us represent the edges of $F(\infty, u/2^{\alpha}p^2)$ as hyperbolic geodesics in the upper half-plane $\mathbb{H}$, that is, as euclidean semi-circles or half-lines perpendicular to real line. Then we have

Theorem 3.2 *$F(\infty, u/2^{\alpha}p^2)$ is self-paired iff $u^2 \equiv -1\,(mod 2^{\alpha-3}p^2)$.*

Proof Because of the transitive action, the form of self-paired edge can be taken as $1/0 \rightarrow u/2^{\alpha}p^2 \rightarrow 1/0$. The condition follows immediately from the second edge by Theorem 3.1.

Now we can give our main theorem. Same problem for modular group was solved in [2] by the same method.

Theorem 3.3 *$F(\infty, u/2^{\alpha}p^2)$ is a forest.*

Proof. Let C be a circuit in $F(\infty, u/2^{\alpha}p^2)$ of minimal length. Suppose first that C is directed, $\nu_1 \stackrel{<}{\longrightarrow} \nu_2 \stackrel{<}{\longrightarrow} ... \stackrel{<}{\longrightarrow} \nu_k$. We may choose the vertices of C apart from ∞ in the interval $[u/2^{\alpha}p^2, (u + 2^{\alpha}p^2)/2^{\alpha}p^2]$ as $\nu_i < \nu_{i+1}$.

We can easily see that $\nu_1 = r/2^{\alpha}p^2$ for $u < r \leq (u + 2^{\alpha}p^2)$ is not possible. Therefore, we take the circuit C as $\infty \rightarrow u/2^{\alpha}p^2 \rightarrow \nu_2 \rightarrow ... \rightarrow \nu_k \rightarrow \infty$. Clearly, $\nu_k > (u+1)/2^{\alpha}p^2$.

Let ν be the largest rational greater than ν_1 for which $\nu_1 \rightarrow \nu$ is an edge in $F(\infty, u/2^{\alpha}p^2)$. We see that ν_2 must equal ν. Assume otherwise that $\nu_2 <$

ν. If ν is a vertex in C, then we obtain a circuit which is a shorter length than C. If ν is not a vertex in C then there are vertices ν_i, ν_{i+1} in C such that $\nu_i < \nu < \nu_{i+1}$. In this case, the edges $\nu_2 \to \nu$ and $\nu_i \to \nu_{i+1}$ cross to each other, it is a contradict the fact that no edges of $F(\infty, u/2^\alpha p^2)$ cross in $\mathbb{H}$. Consequently, $\nu_2 = \nu$. As $\nu_1 < \nu_2$, $\nu_2 = (u + c/d)/2^\alpha p^2$ for some positive integers c and d. Since $\nu_1 \to \nu_2$ is an edge in $F(\infty, u/2^\alpha p^2)$, then $2^\alpha p^2 \nu_1 \to 2^\alpha p^2 \nu_2$ is an edge $F(\infty, u/2^\alpha p^2)$. Thus, c must be 1. From the edge $u/2^\alpha p^2 \to (ud+1)/2^\alpha p^2 d$, we obtain $u^2 + ud + 1 \equiv 0(mod 2^{\alpha-3}p^2)$ by Theorem 3.1. Therefore $\nu_2 = (u+1/d)/2^\alpha p^2$, where d is the smallest positive integer for which $u^2 + ud + 1 \equiv 0(mod 2^{\alpha-3}p^2)$. It is easy to verify that $1 < d < 2^{\alpha-3}p^2$. We define the following transformation

$$\varphi := \begin{pmatrix} -u & (u^2 + ud + 1)/2^\alpha p^2 \\ -2^\alpha p^2 & u + d \end{pmatrix}$$

Then $\varphi \in N_0, \varphi(\infty) = \nu_1, \varphi(\nu_1) = \nu_2$ and, in general, $\varphi((u + x/y)/2^\alpha p^2) = (u + (y/dy - x))/p^2$. φ is increasing on the interval $(-\infty,\ (u + d)/2^\alpha p^2))$, so $\varphi((u+x_1/y_1)/2^\alpha p^2) < \varphi((u+x_2/y_2)/p^2)$ for $x_1/y_1 < x_2/y_2 < d$. Notice that if x and y are positive integers and $x/y < 1$ then $(y/dy - x) < 1$. In fact, since $d \geq 2$ and $y > x$ then $dy - x > y$ and therefore $(y/dy - x) < 1$. Therefore, we can easily see that $\varphi^i(\nu_1) < (u + 1)/2^\alpha p^2$ for positive integers i. We now that $\nu_{i+1} = \varphi^i(\nu_1) = \varphi^{i+1}(\infty)$ for $0 \leq i \leq k - 1$. We already know that $\varphi(\nu_1) = \nu_2$. Now assume that $\nu_i = \varphi^{i-1}(\nu_1)$ for all $1 \leq i \leq s$. Then let us show that $\nu_{s+1} = \varphi^s(\nu_1)$. If not, then first assume that $\nu_{s+1} < \varphi^s(\nu_1)$. Then by transitive action, $\nu_s = \varphi^{s-1}(\nu_1) \to \varphi^{s-1}(\nu_2) = \varphi^s(\nu_1)$ is an edge in $F(\infty, u/2^\alpha p^2)$. If $\varphi^s(\nu_1)$ is not a vertex in C, as $\varphi^s(\nu_1) < \nu_k$, there exist vertices ν_t and ν_{t+1} such that $\nu_t < \varphi^s(\nu_1) < \nu_{t+1}$ and therefore the edges $\nu_t \to \nu_{t+1}$ and $\nu_s \to \varphi^s(\nu_1)$ cross, a contradiction. If $\varphi^s(\nu_1)$ is a vertex in C, as $\nu_{s+1} < \varphi^s(\nu_1), \varphi^s(\nu_1) = \nu_m$ for some $m \geq s + 2$. However, in this case, we would have a circuit $\infty \to \nu_1 \to \nu_2 \to \cdots \to \nu_s \to \nu_k \to \infty$ which is of a shorter length, again a contradiction. Now suppose finally that $\nu_{s+1} > \varphi^s(\nu_1)$. Then from above $\nu_{s+1} > \varphi^s(\nu_1) > \varphi^{s-2}(\nu_1)$ and, as $\varphi^{-(s-1)}(\varphi^{s-2}(\nu_1)) = \infty, \varphi^{-(s-1)}(\nu_{s+1}) > \varphi^{-(s-1)}(\varphi^s(\nu_1)) = \nu_2$. Hence by transitive action, $\nu_1 = \varphi^{-(s-1)}(\nu_s) \to \varphi^{-(s-1)}(\nu_{s+1})$ is an edge in $F(\infty, u/2^\alpha p^2)$, which is contradiction to the choice of ν_2. Consequently $\nu_{i+1} = \varphi^i(\nu_1)$ for $1 \leq i \leq k - 1$. Thus, $\nu_k < (u + 1)/2^\alpha p^2$, a contradiction.

Finally, assume that there is an anti-directed circuit C as minimal length, of the form $\infty \to \nu_1 = u/2^\alpha p^2 \to \cdots \to \nu_t \leftarrow \nu_{t+1} \to \nu_k \to \infty$ for some $t \geq 1$. We know from the above that $\nu_i = \varphi^i(\infty)$ for $i \geq t$. Let ν be the largest rational greater than $u/2^\alpha p^2$ such that $\nu_1 \leftarrow \nu$ is an edge $F(\infty, u/2^\alpha p^2)$. Then $(u + 1/d)/2^\alpha p^2$ for some integer d. By theorem 3.1, $2^{\alpha-3}p^2$ divides d. Since ν is the largest we have $2^{\alpha-3}p^2 = d$. Thus $\nu_2 \leq \nu = (u + 1/d)/2^\alpha p^2$. As $d < 2^{\alpha-3}p^2$ then $u < (u + 1/d)/2^\alpha p^2$. Hence t must be greater than 1,

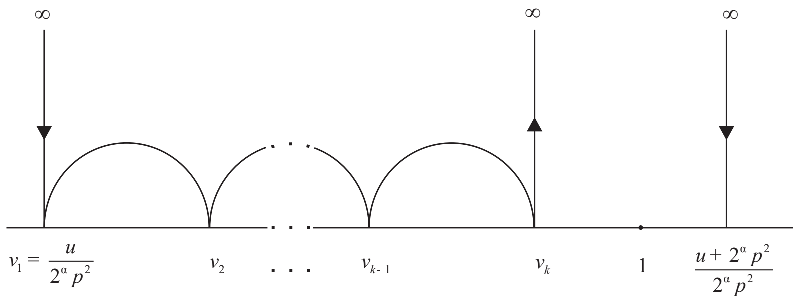

Figure 2: Path of the action

otherwise $\nu_s = (u+1/d)/2^\alpha p^2$ for some $s \geq 3$ and then we would circuit $\infty \to \nu_1 \to \nu_s \cdots \to \nu_k \to \infty$ of a shorter length, a contradiction. Hence we must have $\nu_1 \to \nu_2 = (u+1/d)/2^\alpha p^2$ Let $\omega = \varphi^{t+1}(\infty)$. Since, by transitive action, $\nu_t = \varphi^{t-1}(\nu_1) \to \varphi^{t-1}(\nu_2 = \omega)$ is an edge $F(\infty, u/2^\alpha p^2)$ we see that $\nu_{t+1} \neq \omega$, otherwise, by Theorem 3.1, $\nu_t \leftarrow \nu_{t+1}$ and $\nu_t \to \nu_{t+1}$ imply that $u^2 \equiv -1(mod 2^{\alpha-3}p^2)$ which, as $d < 2^{\alpha-3}p^2$, is a contradiction to $u^2 + ud + 1 \equiv 0(mod 2^{\alpha-3}p^2)$. Therefore, the inequality $\nu_{t+1} < \omega$ must be true. For if $\nu_{t+1} > \omega$, then, as $\varphi^{-(t-1)}(\varphi^{t-2}(\nu_1)) = \infty$ and $\varphi^{t-2}(\nu_1) < \varphi^t(\nu_1) = \omega < \nu_{t+1}$, $\varphi^{-(t-1)}(\omega) = \nu_2$ and $\varphi^{-(t-1)}(\nu_t) = \nu_1 \leftarrow \varphi^{-(t-1)}(\nu_{t+1}) > \nu_2$ is an edge in $F(\infty, u/2^\alpha p^2)$, which contradicts the choice of ν_2. However, if $\nu_{t+1} < \omega$ then we would have $\omega = \nu_s$ for some $s \geq t+2$ and therefore we would have the circuit $\infty \to \nu_1 \to \cdots \to \nu_t \to \nu_s \to \cdots \to \nu_k \to \infty$ of a shorter length, which again gives a contradiction. This shows that C must be directed. Hence the proof of the theorem is completed. ▪

At this point, situation seems to be as following;

Conjecture. Let N have the prime power decomposition as $2^\alpha \cdot 3^\beta \cdot p_3^{\gamma_3} \cdots p_r^{\gamma_r}$. Among others than the case of the transitive action, also for $\beta \geq 4$, the suborbital graphs of normalizer would be a forest.

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