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Leonardo Cartan Numbers and Related Fibonacci–Lucas Structures

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ABSTRACT

This paper investigates the Leonardo Cartan numbers, defined as an extension of the classical Leonardo sequence through additional algebraic structures. The recurrence relations of these numbers are established, and various summation formulas are derived. Furthermore, key mathematical identities, including Catalan's and Cassini's identities, are obtained within the framework of Leonardo Cartan numbers. The relationships among Fibonacci, Lucas, and Leonardo sequences are also explored, thereby highlighting their interconnections. By employing Binet's formula, several fundamental properties of Leonardo Cartan numbers are confirmed, contributing to a deeper understanding of their algebraic structure.

1 | Introduction

The pursuit of new algebraic systems that unify recursive integer sequences with multidimensional hypercomplex structures has become one of the most stimulating topics in modern mathematics. Four-dimensional algebras such as quaternions and their split counterparts provide a natural bridge between algebra, geometry, and mathematical physics. Beyond their classical applications in representing rotations and Lorentzian transformations, these algebras allow the embedding of well-known recurrence sequences into richer algebraic frameworks, thereby revealing deeper structural connections and computational properties. In this broader context, the integration of Fibonacci-type sequences with quaternionic or Clifford-like algebras has emerged as a vibrant research direction.

In this regard, Flaut and collaborators have made significant contributions. Flaut and Savin [1, 2] developed quaternionic and Clifford algebraic representations of generalized Fibonacci–Lucas sequences, offering a unified view of their algebraic behavior. Later, Flaut, Shpakivskyi, and Vlad [3] extended these

constructions to arbitrary algebras using $h(x)$ -Fibonacci polynomials, while Flaut, Savin, and Zaharia [4] explored computational and algorithmic applications of Fibonacci and Lucas numbers in modern applied mathematics. Parallel to these developments, Szynal-Liana and Włoch [5, 6], Bród et al. [7], and Costa et al. [8] introduced novel Fibonacci-type hypercomplex systems, including hybrid, bihyperbolic, and tricomplex extensions, thereby highlighting the growing interplay between recurrence relations, generalized quaternionic structures, and multidimensional algebraic analysis. In addition, Kızılateş and Kone [9] studied higher order Fibonacci hypercomplex numbers, further enriching the theory of hypercomplex extensions of classical integer sequences. Furthermore, Kızılateş and Yachani [10] investigated higher order Gaussian Fibonacci quaternions, while Yılmaz et al. [11] introduced generalized Pauli Fibonacci polynomial quaternions, contributing to the algebraic generalization of Fibonacci-type structures. Furthermore, Polatlı [12] studied hybrid numbers with Fibonacci and Lucas coefficients, and Çolak et al. [13] developed generalized Fibonacci polynomial spinors, extending the theory to new algebraic settings. In

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a related context, Panwar [14] examined identities of bivariate Fibonacci and Lucas polynomials, further enriching the structural diversity of Fibonacci-type sequences.

In the context of Leonardo-type sequences, Atasoy [15] introduced a new polar representation for split Leonardo quaternions and derived several identities, demonstrating the extension of these sequences within quaternionic algebraic frameworks. Moreover, Beites and Catarino [16] investigated the Leonardo quaternion sequence and established various algebraic properties and identities, contributing to the development of quaternionic formulations.

More recently, Yağmur [17] examined higher order Leonardo hypercomplex numbers, further extending these sequences within generalized hypercomplex algebraic settings.

Building on these developments, Öztürk [18] introduced the concept of Cartan numbers, a four-dimensional algebra isomorphic to the split quaternion algebra, combining algebraic flexibility with geometric interpretability. The subsequent extension to dual Cartan numbers [19] enriched this framework by connecting it with pseudo-null lines in Lorentzian geometry. Within this setting, Cartan numbers provide a robust algebraic framework for constructing new number sequences that preserve associativity while facilitating novel hypercomplex identities.

The algebra of Cartan numbers, denoted by $\mathbb{S}$, is defined as

$$\mathbb{S} = \{\mathbf{q} = q_1 + q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k} : q_r \in \mathbb{R}, \mathbf{i}^2 = 1, \mathbf{j}^2 = \mathbf{k}^2 = 0, \mathbf{j}\mathbf{k} = 1 + \mathbf{i}, \mathbf{k}\mathbf{j} = 1 - \mathbf{i}\}. \quad (1)$$

Let $\mathbf{p} = p_1 + p_2\mathbf{i} + p_3\mathbf{j} + p_4\mathbf{k}$ and $\mathbf{q} = q_1 + q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k}$ be arbitrary elements of $\mathbb{S}$. Their fundamental operations are defined as follows:

1. Equality: $\mathbf{p} = \mathbf{q}$ if and only if $p_r = q_r$ for $r = 1, 2, 3, 4$.
2. Addition and subtraction: $\mathbf{p} \pm \mathbf{q} = (p_1 \pm q_1) + (p_2 \pm q_2)\mathbf{i} + (p_3 \pm q_3)\mathbf{j} + (p_4 \pm q_4)\mathbf{k}$.
3. Scalar multiplication: For any $\lambda \in \mathbb{R}$, $\lambda\mathbf{p} = \lambda p_1 + \lambda p_2\mathbf{i} + \lambda p_3\mathbf{j} + \lambda p_4\mathbf{k}$.

The multiplication of the basis elements is shown in Table 1.

Cartan numbers are commutative and associative under addition, and associative but noncommutative under multiplication. For $\mathbf{q} \in \mathbb{S}$, its conjugate and character are defined, respectively, as

$$\begin{aligned} \bar{\mathbf{q}} &= q_1 - q_2\mathbf{i} - q_3\mathbf{j} - q_4\mathbf{k}, \\ C(\mathbf{q}) &= \mathbf{q}\bar{\mathbf{q}} = q_1^2 - q_2^2 - 2q_3q_4. \end{aligned} \quad (2)$$

Further details on Cartan numbers and their geometric significance can be found in [18, 19].

In the literature, integer sequences occupy a central position. Among these, the Fibonacci and Lucas sequences are among the most well-known and widely studied. Under the initial conditions $F_0 = 0, F_1 = 1$ and $L_0 = 2, L_1 = 1$, the Fibonacci numbers F_n and Lucas numbers L_n satisfy

$$F_n = F_{n-1} + F_{n-2}, \quad (3)$$

and

$$L_n = L_{n-1} + L_{n-2}, \quad (4)$$

TABLE 1 | Multiplication table of Cartan number units.

$\cdot$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	1	$\mathbf{j}$	$-\mathbf{k}$
$\mathbf{j}$	$-\mathbf{j}$	0	$1 + \mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$1 - \mathbf{i}$	0

whose characteristic equation is $\lambda^2 - \lambda - 1 = 0$. Their Binet's formulas are

$$F_n = \frac{\sigma^n - \rho^n}{\sigma - \rho}, \quad (5)$$

and

$$L_n = \sigma^n + \rho^n, \quad (6)$$

where σ and ρ denote the roots of the characteristic equation.

This study focuses on investigating an alternative sequence that exhibits similarities to the Fibonacci sequence, known as the Leonardo sequence, denoted by Le_n . Initial studies on Leonardo numbers were conducted by Catarino and Borges [20, 21], where their fundamental properties were established. The Leonardo sequence follows a recurrence relation similar to that of the Fibonacci sequence but includes an additional constant term, which leads to a distinct growth behavior:

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \quad n \geq 2, \quad Le_0 = Le_1 = 1. \quad (7)$$

A related relation between consecutive terms is

$$Le_{n+1} = 2Le_n - Le_{n-2}, \quad (8)$$

whose characteristic equation is

$$\lambda^3 - 2\lambda^2 + 1 = 0. \quad (9)$$

Binet's formula for the Leonardo sequence is

$$Le_n = \frac{2\sigma^{n+1} - 2\rho^{n+1}}{\sigma - \rho} - 1, \quad (10)$$

where $\sigma = (1 + \sqrt{5})/2$ and $\rho = (1 - \sqrt{5})/2$. It follows that

$$Le_n = 2F_{n+1} - 1, \quad (11)$$

where F_n is the n th Fibonacci number.

Catarino and Borges, using this identity, derived the generating function of Leonardo numbers and established the Cassini, Catalan, and d'Ocagne identities [20]. Shannon [22] defined generalized Leonardo numbers as an extension of Asveld's and Horadam's sequences. Vieira et al. [23] analyzed the two-

dimensional recurrence relations of Leonardo numbers. Incomplete Leonardo numbers were defined in [21]. Alp and Koçer [24, 25] introduced the matrix representation of Leonardo numbers and proposed hybrid Leonardo numbers, examining their algebraic properties. In addition, Soykan [26] investigated generalized Leonardo numbers, while Gökbaşı [27] introduced k -Leonardo numbers, providing further extensions of the classical recurrence structure. Moreover, Dışkaya et al. [28] examined hyperbolic Leonardo and hyperbolic Francois quaternions, highlighting the interaction between Leonardo-type sequences and hypercomplex algebraic systems.

Fibonacci, Lucas, and Leonardo numbers satisfy various mathematical identities, such as

$$F_n + L_n = 2F_{n+1}, \quad (12)$$

$$L_{n+2} + L_n = 5F_{n+1}, \quad (13)$$

$$L_{m+n} - (-1)^n L_{m-n} = 5F_m F_n, \quad (14)$$

$$F_{m+n} + (-1)^n F_{m-n} = F_m L_n, \quad (15)$$

$$L_{2k} = 5F_k^2 + 2(-1)^k, \quad (16)$$

$$Le_{n+m} + (-1)^m Le_{n-m} = L_m(Le_n + 1) - 1 - (-1)^m, \quad (17)$$

$$Le_{n+m} - (-1)^m Le_{n-m} = L_{n+1}(Le_{m-1} + 1) - 1 + (-1)^m, \quad (18)$$

$$\sum_{s=1}^r (-1)^{s-1} F_{s+1} = (-1)^{r-1} F_r, \quad (19)$$

$$F_{r+s} - (-1)^s F_{r-s} = F_s L_r. \quad (20)$$

For further properties of Fibonacci and Lucas numbers, see [20, 24, 29, 30].

Despite the increasing number of studies on hypercomplex extensions of Fibonacci-type sequences, there is no systematic framework that incorporates Leonardo numbers into the algebraic structure of Cartan numbers. In particular, although quaternionic and hypercomplex formulations of Leonardo-type sequences have been investigated, their integration within Cartan algebra, which is isomorphic to the split quaternion algebra, has not been addressed in the existing literature.

It is worth noting that hypercomplex extensions of recurrence sequences have been considered in various algebraic settings. However, the aim of the present work is not to reproduce such constructions in another algebraic system but to provide a systematic development within the Cartan algebra.

In particular, the relation $CLe_n = 2CF_{n+1} - I$ serves as a central structural link connecting Leonardo Cartan numbers to Fibonacci Cartan numbers. This allows the theory to be organized around a single underlying principle, rather than a collection of independent identities.

In order to address this gap, the present study aims to develop a new algebraic framework by introducing Leonardo Cartan numbers. The main contributions of this paper can be summarized as follows: (i) the definition of Leonardo Cartan numbers within the Cartan algebra structure, (ii) the derivation of

Binet-type formulas and generating functions, and (iii) the extension of classical identities such as Catalan and Cassini identities to this new setting.

Building on these foundations, this paper introduces the concept of Leonardo Cartan numbers, a novel hypercomplex extension that integrates the recurrence properties of Leonardo numbers within the algebraic framework of Cartan numbers. This construction unifies the structures of Fibonacci, Lucas, and Leonardo sequences under the Cartan algebra, leading to new formulations of generating functions, Binet-type expressions, and classical identities such as the Catalan and Cassini identities.

The remainder of the paper is organized as follows. Section 2 introduces the Leonardo Cartan numbers and presents their fundamental properties. Section 3 provides concluding remarks.

2 | Leonardo Cartan Numbers

In this section, the formal definition of Leonardo Cartan numbers is presented. Based on this definition, several fundamental mathematical expressions are established, including the generating function, Binet's formula, summation formulas, and key identities such as the Catalan and Cassini identities, as well as additional related properties.

Several results presented in this section follow a similar structural pattern and are included for completeness. Many of these can also be interpreted as direct consequences of the main identities.

Definition 1. For any integer $n \geq 0$, Leonardo Cartan number $\{CLe_i\}_{i=0}^\infty$ is defined as

$$CLe_n = Le_n + Le_{n+1}\mathbf{i} + Le_{n+2}\mathbf{j} + Le_{n+3}\mathbf{k}, \quad (21)$$

where Le_n is the n -th Leonardo number.

For clarity, Leonardo Cartan numbers are denoted by CLe_n throughout the paper, in order to distinguish them from Fibonacci Cartan numbers CF_n and Lucas Cartan numbers CL_n . The first few Leonardo Cartan numbers are given as follows:

$$\begin{aligned} CLe_0 &= 1 + \mathbf{i} + 3\mathbf{j} + 5\mathbf{k}, \\ CLe_1 &= 1 + 3\mathbf{i} + 5\mathbf{j} + 9\mathbf{k}, \\ CLe_2 &= 3 + 5\mathbf{i} + 9\mathbf{j} + 15\mathbf{k}, \\ CLe_3 &= 5 + 9\mathbf{i} + 15\mathbf{j} + 25\mathbf{k}. \end{aligned} \quad (22)$$

Using the recurrence relations presented in Equations (7) and (21), the following expression is obtained:

$$\begin{aligned} CLe_n &= Le_n + Le_{n+1}\mathbf{i} + Le_{n+2}\mathbf{j} + Le_{n+3}\mathbf{k} \\ &= (Le_{n-1} + Le_{n-2} + 1) + (Le_n + Le_{n-1} + 1)\mathbf{i} \\ &\quad + (Le_{n+1} + Le_n + 1)\mathbf{j} + (Le_{n+2} + Le_{n+1} + 1)\mathbf{k} \\ &= CLe_{n-1} + CLe_{n-2} + I, \end{aligned} \quad (23)$$

where $n \geq 2$ and $I = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$. It is important to note that from this point forward, the expression $1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$ will consistently be represented as I throughout the paper. From the recurrence relations for the Leonardo numbers, as defined by Equations (8) and (21), the following solution is obtained:

$$\begin{aligned} CLe_{n+1} &= Le_{n+1} + Le_{n+2}\mathbf{i} + Le_{n+3}\mathbf{j} + Le_{n+4}\mathbf{k} \\ &= (2Le_n - Le_{n-2}) + (2Le_{n+1} - Le_{n-1})\mathbf{i} \\ &\quad + (2Le_{n+2} - Le_n)\mathbf{j} + (2Le_{n+3} - Le_{n+1})\mathbf{k} \\ &= 2CLe_n - CLe_{n-2}, \end{aligned} \quad (24)$$

for $n \geq 2$. Consequently, the following additional recurrence relation for Leonardo Cartan numbers is obtained:

$$CLe_{n+1} = 2CLe_n - CLe_{n-2}, \quad \text{for } n \geq 2. \quad (25)$$

Theorem 1. The generating function for the Leonardo Cartan numbers $\{CLe_i\}_{i=0}^{\infty}$ is

$$\begin{aligned} f(x) &= \sum_{i=0}^{\infty} CLe_i x^i \\ &= \frac{CLe_0 + (-1 + \mathbf{i} - \mathbf{j} - \mathbf{k})x + (1 - \mathbf{i} - \mathbf{j} - 3\mathbf{k})x^2}{1 - 2x + x^3}. \end{aligned} \quad (26)$$

Proof 1. Since the generating function for the sequence $\{CLe_i\}_{i=0}^{\infty}$ is represented by the formal power series

$$f(x) = \sum_{i=0}^{\infty} CLe_i x^i, \quad (27)$$

it follows that

$$f(x) = CLe_0 + CLe_1 x + CLe_2 x^2 + \cdots + CLe_n x^n + \cdots. \quad (28)$$

Thus,

$$\begin{aligned} 2xf(x) &= 2CLe_0 x + 2CLe_1 x^2 + \cdots + 2CLe_n x^{n+1} + \cdots, \\ x^3 f(x) &= CLe_0 x^3 + CLe_1 x^4 + \cdots + CLe_n x^{n+3} + \cdots. \end{aligned} \quad (29)$$

It follows from the above equations that

$$\begin{aligned} (1 - 2x + x^3)f(x) &= CLe_0 + (CLe_1 - 2CLe_0)x \\ &\quad + (CLe_2 - 2CLe_1)x^2 + \cdots \\ &\quad + (CLe_{n+1} - 2CLe_n + CLe_{n-2})x^{n+1} + \cdots. \end{aligned} \quad (30)$$

From the recurrence relation $CLe_{n+1} = 2CLe_n - CLe_{n-2}$, it follows that

$$f(x) = \frac{CLe_0 + (-1 + \mathbf{i} - \mathbf{j} - \mathbf{k})x + (1 - \mathbf{i} - \mathbf{j} - 3\mathbf{k})x^2}{1 - 2x + x^3}. \quad (31)$$

So the proof is completed. $\square$

Let us formulate the equation that establishes a relationship between the Leonardo Cartan numbers and the Fibonacci Cartan numbers introduced by Öztürk and Çakır [31].

Theorem 2. For any nonnegative integer n , the following relation holds

$$CLe_n = 2CF_{n+1} - I, \quad (32)$$

where CF_n denotes the n -th Fibonacci Cartan number.

Proof 2. From Equation (11) and the recurrence relations of Fibonacci Cartan and Leonardo Cartan numbers, it follows that

$$\begin{aligned} CLe_n &= Le_n + Le_{n+1}\mathbf{i} + Le_{n+2}\mathbf{j} + Le_{n+3}\mathbf{k} \\ &= (2F_{n+1} - 1) + (2F_{n+2} - 1)\mathbf{i} + (2F_{n+3} - 1)\mathbf{j} \\ &\quad + (2F_{n+4} - 1)\mathbf{k} \\ &= 2(F_{n+1} + F_{n+2}\mathbf{i} + F_{n+3}\mathbf{j} + F_{n+4}\mathbf{k}) - (1 + \mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &= 2CF_{n+1} - I. \end{aligned} \quad (33)$$

Substituting (12) into (32) yields the following expression:

$$CLe_n = CF_n + CL_n - I, \quad (34)$$

where CF_n denotes the n -th Fibonacci Cartan number, and CL_n represents the n -th Lucas Cartan number, as stated in [31].

Next, the Binet's formula for Leonardo Cartan numbers is given.

Theorem 3. For any nonnegative integer n , the Binet's formula for the Leonardo Cartan numbers CLe_n is given by

$$CLe_n = \frac{2\hat{\sigma}\sigma^{n+1} - 2\hat{\rho}\rho^{n+1}}{\sigma - \rho} - I, \quad (35)$$

where σ and ρ represent the roots of the characteristic equation (9), defined as $\hat{\sigma} = 1 + \sigma\mathbf{i} + \sigma^2\mathbf{j} + \sigma^3\mathbf{k}$ and $\hat{\rho} = 1 + \rho\mathbf{i} + \rho^2\mathbf{j} + \rho^3\mathbf{k}$.

Proof 3. The Binet's formula for Fibonacci Cartan numbers is known to be

$$CF_{n+1} = \frac{\hat{\sigma}\sigma^{n+1} - \hat{\rho}\rho^{n+1}}{\sigma - \rho}, \quad (36)$$

where CF_n denotes the n -th Fibonacci Cartan number, as defined in [31]. The result follows directly from Equation (32), which establishes the relation between Fibonacci Cartan and Leonardo Cartan numbers. $\square$

Next, some results concerning the sums of terms in the Leonardo Cartan sequence are presented, employing known summation formulas for Leonardo numbers.

Theorem 4. For any nonnegative integer n , the summation formulas for the Leonardo Cartan numbers are as follows:

$$\begin{aligned} \sum_{i=0}^n CLe_i &= CLe_{n+2} - (2CF_2 + nI), \\ \sum_{i=0}^n CLe_{2i} &= CLe_{2n+1} - (2CF_0 + nI), \\ \sum_{i=0}^n CLe_{2i+1} &= CLe_{2n+2} - (2CF_1 + nI). \end{aligned} \quad (37)$$

Additionally, for $n \geq 1$, the following relations hold

$$\sum_{i=0}^n (-1)^i CLe_i = \begin{cases} I - 2\mathbf{i} - CLe_{n-1}, & n \text{ is odd,} \\ CLe_{n-1} + 2(I - \mathbf{i}), & n \text{ is even.} \end{cases} \quad (38)$$

Proof 4. From Binet's formula for Leonardo Cartan numbers, the following expressions are obtained sequentially. Since this formula is given by

$$CLe_i = \frac{2(\hat{\sigma}\sigma^{i+1} - \hat{\rho}\rho^{i+1})}{\sigma - \rho} - I = CL_1 F_i + CF_1 L_i - I, \quad (39)$$

the first summation identity can be derived as follows:

$$\begin{aligned}\sum_{i=0}^n CLe_i &= CL_1 \sum_{i=0}^n F_i + CF_1 \sum_{i=0}^n L_i - \sum_{i=0}^n I \\ &= CL_1(F_{n+2} - 1) + CF_1(L_{n+2} - 1) - (n+1)I \\ &= CLe_{n+2} - (2CF_2 + nI).\end{aligned}\quad (40)$$

The second summation identity is derived as

$$\begin{aligned}\sum_{i=0}^n CLe_{2i} &= CL_1 \sum_{i=0}^n F_{2i} + CF_1 \sum_{i=0}^n L_{2i} - \sum_{i=0}^n I \\ &= CL_1(F_{2n+1} - 1) + CF_1(L_{2n+1} + 1) - (n+1)I \\ &= CLe_{2n+1} - (2CF_0 + nI).\end{aligned}\quad (41)$$

Similarly, the third summation identity is derived as

$$\begin{aligned}\sum_{i=0}^n CLe_{2i+1} &= CL_1 \sum_{i=0}^n F_{2i+1} + CF_1 \sum_{i=0}^n L_{2i+1} - \sum_{i=0}^n I \\ &= CL_1 F_{2n+2} + CF_1(L_{2n+2} - 2) - (n+1)I \\ &= CLe_{2n+2} - (2CF_1 + nI).\end{aligned}\quad (42)$$

Now, let us prove the final summation identity. If n is odd, then

$$\begin{aligned}\sum_{i=0}^n (-1)^i CLe_i &= \sum_{i=0}^{(n-1)/2} CLe_{2i} - \sum_{i=0}^{(n-1)/2} CLe_{2i+1} \\ &= CLe_n - 2CF_0 - \left(\frac{n-1}{2}\right)I \\ &\quad - CLe_{n+1} + 2CF_1 + \left(\frac{n-1}{2}\right)I \\ &= CLe_n - CLe_{n+1} + 2(CF_1 - CF_0) \\ &= I - 2\mathbf{i} - CLe_{n-1},\end{aligned}\quad (43)$$

and if n is even,

$$\begin{aligned}\sum_{i=0}^n (-1)^i CLe_i &= \sum_{i=0}^{n/2} CLe_{2i} - \sum_{i=0}^{(n-2)/2} CLe_{2i+1} \\ &= CLe_{n+1} - 2CF_0 - \frac{n}{2}I - CLe_n + 2CF_1 \\ &\quad + \left(\frac{n-2}{2}\right)I \\ &= CLe_{n+1} - CLe_n - I - 2CF_0 + 2CF_1 \\ &= CLe_{n-1} + 2(I - \mathbf{i}).\end{aligned}\quad (44)$$

□

Among the most well-known identities associated with Fibonacci numbers are Catalan's and Cassini's identities. Owing to the intrinsic relationship between Fibonacci and Leonardo numbers, these identities extend naturally to Leonardo numbers. Consequently, the corresponding identities for Leonardo Cartan numbers can be derived as follows.

Theorem 5 (Catalan's Identity). *For positive integers n and r , where $n \geq r$, the following identity holds*

$$\begin{aligned}CLe_n^2 - CLe_{n-r}CLe_{n+r} &= (CLe_{n-r} - CLe_n)I + I(CLe_{n+r} - CLe_n) \\ &\quad + 4(-1)^{n-r+1}(F_r^2(CL_0 - 1) \\ &\quad + F_r L_r(\mathbf{i} - \mathbf{j} + \mathbf{k})),\end{aligned}\quad (45)$$

where F_n denotes the n -th Fibonacci number.

Proof 5. Let σ and ρ be the roots of the characteristic Equation (9), and we define $\hat{\sigma} = 1 + \sigma\mathbf{i} + \sigma^2\mathbf{j} + \sigma^3\mathbf{k}$ and $\hat{\rho} = 1 + \rho\mathbf{i} + \rho^2\mathbf{j} + \rho^3\mathbf{k}$. Then, the following identities hold

$$\hat{\sigma}\hat{\rho} = 1 + (1 - \sqrt{5})\mathbf{i} + (3 + \sqrt{5})\mathbf{j} + (4 - \sqrt{5})\mathbf{k}, \quad (46)$$

$$\hat{\rho}\hat{\sigma} = 1 + (1 + \sqrt{5})\mathbf{i} + (3 - \sqrt{5})\mathbf{j} + (4 + \sqrt{5})\mathbf{k}. \quad (47)$$

Applying Binet's formula (35) to the left-hand side (LHS) of Equation (45) and substituting Equations (46) and (47) into the Binet representation yield the following identity:

$$\begin{aligned}\text{LHS} &= \left(\frac{2\hat{\sigma}\sigma^{n+1} - 2\hat{\rho}\rho^{n+1} - I\sqrt{5}}{\sigma - \rho}\right) \left(\frac{2\hat{\sigma}\sigma^{n+1} - 2\hat{\rho}\rho^{n+1} - I\sqrt{5}}{\sigma - \rho}\right) \\ &\quad - \left(\frac{2\hat{\sigma}\sigma^{n-r+1} - 2\hat{\rho}\rho^{n-r+1} - I\sqrt{5}}{\sigma - \rho}\right) \left(\frac{2\hat{\sigma}\sigma^{n+r+1}}{\sigma - \rho}\right) \\ &\quad + \frac{-2\hat{\rho}\rho^{n+r+1} - I\sqrt{5}}{\sigma - \rho} \\ &= \left[\left(\frac{2\hat{\sigma}\sigma^{n-r+1} - 2\hat{\rho}\rho^{n-r+1}}{\sigma - \rho}\right) - \left(\frac{2\hat{\sigma}\sigma^{n+1} - 2\hat{\rho}\rho^{n+1}}{\sigma - \rho}\right)\right]I \\ &\quad + I\left[\left(\frac{2\hat{\sigma}\sigma^{n+r+1} - 2\hat{\rho}\rho^{n+r+1}}{\sigma - \rho}\right) - \left(\frac{2\hat{\sigma}\sigma^{n+1} - 2\hat{\rho}\rho^{n+1}}{\sigma - \rho}\right)\right] \\ &\quad + \frac{4\hat{\sigma}\hat{\rho}\sigma^{n-r+1}\rho^{n+r+1}}{(\sigma - \rho)^2} + \frac{4\hat{\rho}\hat{\sigma}\rho^{n-r+1}\sigma^{n+r+1}}{(\sigma - \rho)^2} \\ &\quad - \frac{4\hat{\sigma}\hat{\rho}(-1)^{n+1}}{(\sigma - \rho)^2} - \frac{4\hat{\rho}\hat{\sigma}(-1)^{n+1}}{(\sigma - \rho)^2} \\ &= (CLe_{n-r} - CLe_n)I + I(CLe_{n+r} - CLe_n) \\ &\quad + 4(-1)^{n+1}\left(\frac{\hat{\sigma}\hat{\rho}\sigma^{-r}\rho^r + \hat{\rho}\hat{\sigma}\rho^{-r}\sigma^r}{(\sigma - \rho)^2} - \frac{2 + 2\mathbf{i} + 6\mathbf{j} + 8\mathbf{k}}{5}\right).\end{aligned}\quad (48)$$

Here, by using the Binet representation of CLe_n given in (35), the first two bracketed terms reduce to $(CLe_{n-r} - CLe_n)$ and $(CLe_{n+r} - CLe_n)$, respectively.

$$\begin{aligned}&= (CLe_{n-r} - CLe_n)I + I(CLe_{n+r} - CLe_n) \\ &\quad + 4(-1)^{n+1}\left[\frac{L_{2r}(-1)^r}{5}(1 + \mathbf{i} + 3\mathbf{j} + 4\mathbf{k})\right. \\ &\quad \left.+ F_{2r}(-1)^r(\mathbf{i} - \mathbf{j} + \mathbf{k}) - \frac{2 + 2\mathbf{i} + 6\mathbf{j} + 8\mathbf{k}}{5}\right].\end{aligned}\quad (49)$$

From the identities $L_{2r} = 5F_r^2 + 2(-1)^r$ in (16) and $F_{2r} = F_r L_r$, obtained from (20) by setting $s = r$, the following result is obtained:

$$\begin{aligned}\text{LHS} &= (CLe_{n-r} - CLe_n)I + I(CLe_{n+r} - CLe_n) \\ &\quad + 4(-1)^{n+1}\left[\frac{5F_r^2(-1)^r + 2(-1)^r(-1)^r}{5}(1 + \mathbf{i} + 3\mathbf{j} + 4\mathbf{k})\right. \\ &\quad \left.+ F_r L_r(-1)^r(\mathbf{i} - \mathbf{j} + \mathbf{k}) - \frac{2 + 2\mathbf{i} + 6\mathbf{j} + 8\mathbf{k}}{5}\right].\end{aligned}\quad (50)$$

Note that

$$CL_0 = 2 + \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}, \quad (51)$$

and hence,

$$CL_0 - 1 = 1 + \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}, \quad (52)$$

is used in the simplification.

$$\begin{aligned} &= (CLE_{n-r} - CLe_n)I + I(CLe_{n+r} - CLe_n) \\ &\quad + 4(-1)^{n+1} \left[F_r^2(-1)^r(CL_0 - 1) + \frac{2(1 + \mathbf{i} + 3\mathbf{j} + 4\mathbf{k})}{5} \right. \\ &\quad \left. + F_r L_r(-1)^r(\mathbf{i} - \mathbf{j} + \mathbf{k}) - \frac{2 + 2\mathbf{i} + 6\mathbf{j} + 8\mathbf{k}}{5} \right] \\ &= (CLE_{n-r} - CLe_n)I + I(CLe_{n+r} - CLe_n) \\ &\quad + 4(-1)^{n-r+1} [F_r^2(CL_0 - 1) + F_r L_r(\mathbf{i} - \mathbf{j} + \mathbf{k})]. \end{aligned} \quad (53)$$

Thus, Equation (45) is obtained. $\square$

To illustrate the validity of Catalan's identity given in Theorem 25, we consider the case $n = 5$ and $r = 2$.

Using the definition of Leonardo Cartan numbers, we obtain

$$\begin{aligned} CLe_3 &= 5 + 9\mathbf{i} + 15\mathbf{j} + 25\mathbf{k}, \\ CLe_5 &= 15 + 25\mathbf{i} + 41\mathbf{j} + 67\mathbf{k}, \\ CLe_7 &= 41 + 67\mathbf{i} + 109\mathbf{j} + 177\mathbf{k}. \end{aligned} \quad (54)$$

First, we compute the LHS:

$$CLe_5^2 - CLe_3 CLe_7. \quad (55)$$

Using the multiplication rules given in Table 1, we obtain

$$\begin{aligned} CLe_5^2 &= 6344 + 750\mathbf{i} + 1230\mathbf{j} + 2010\mathbf{k}, \\ CLe_3 CLe_7 &= 6188 + 634\mathbf{i} + 1136\mathbf{j} + 1992\mathbf{k}. \end{aligned} \quad (56)$$

Thus,

$$CLe_5^2 - CLe_3 CLe_7 = 156 + 116\mathbf{i} + 94\mathbf{j} + 18\mathbf{k}. \quad (57)$$

Now, we consider the right-hand side of Equation (45):

$$\begin{aligned} &(CLe_3 - CLe_5)I + I(CLe_7 - CLe_5) \\ &\quad + 4(-1)^{5-2+1} (F_2^2(CL_0 - 1) + F_2 L_2(\mathbf{i} - \mathbf{j} + \mathbf{k})). \end{aligned} \quad (58)$$

Using $F_2 = 1$, $L_2 = 3$, and $CL_0 = 2 + \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, straightforward computation yields

$$156 + 116\mathbf{i} + 94\mathbf{j} + 18\mathbf{k}. \quad (59)$$

Since both sides coincide, the identity is verified for $n = 5$ and $r = 2$.

Setting $r = 1$ in Equation (45) yields the Cassini-like identity for Leonardo Cartan numbers, stated in the following corollary.

Corollary 1 (Cassini's Identity). *For $n \geq 1$, the following identity is satisfied:*

$$CLe_n^2 - CLe_{n-1} CLe_{n+1} = I CLe_{n-1} - CLe_{n-2} I + 4(-1)^n (CF_2 - \mathbf{j}). \quad (60)$$

Similarly, the Cassini identity can be illustrated by considering $n = 2$. Using the values of CLe_1 , CLe_2 , and CLe_3 , it follows by direct computation that both sides of the corresponding identity are equal, confirming the result.

Theorem 6. *For any positive integers n, r , and s , the following identity holds*

$$\begin{aligned} CLe_{n+r} CLe_{n+s} - CLe_n CLe_{n+r+s} &= (CLe_n - CLe_{n+r})I \\ &\quad + I(CLe_{n+r+s} - CLe_{n+s}) \\ &\quad + 4(-1)^{n+1} F_r(F_s(CL_0 - 1) \\ &\quad + L_s(\mathbf{i} - \mathbf{j} + \mathbf{k})), \end{aligned} \quad (61)$$

where F_n and L_n denote the n -th Fibonacci and Lucas numbers, respectively.

Proof 6. Applying Binet's formula (35) to the LHS of Equation (61) and substituting Equations (46) and (47) into the Binet representation yield the following identity:

$$\begin{aligned} \text{LHS} &= \left(\frac{2\hat{\sigma}\sigma^{n+r+1} - 2\hat{\rho}\rho^{n+r+1}}{\sigma - \rho} - I \right) \left(\frac{2\hat{\sigma}\sigma^{n+s+1} - 2\hat{\rho}\rho^{n+s+1}}{\sigma - \rho} - I \right) \\ &\quad - \left(\frac{2\hat{\sigma}\sigma^{n+1} - 2\hat{\rho}\rho^{n+1}}{\sigma - \rho} - I \right) \left(\frac{2\hat{\sigma}\sigma^{n+r+s+1}}{\sigma - \rho} \right. \\ &\quad \left. - \frac{2\hat{\rho}\rho^{n+r+s+1}}{\sigma - \rho} - I \right) \\ &= \left[\frac{2\hat{\sigma}\sigma^{n+1}I - 2\hat{\rho}\rho^{n+1}I}{\sigma - \rho} - \frac{2\hat{\sigma}\sigma^{n+r+1}I - 2\hat{\rho}\rho^{n+r+1}I}{\sigma - \rho} \right. \\ &\quad \left. + \frac{I2\hat{\rho}\rho^{n+s+1} - I2\hat{\sigma}\sigma^{n+s+1}}{\sigma - \rho} + \frac{I2\hat{\sigma}\sigma^{n+r+s+1} - I2\hat{\rho}\rho^{n+r+s+1}}{\sigma - \rho} \right. \\ &\quad \left. - \frac{(-1)^n (-4\hat{\rho}\hat{\sigma}\rho^r\sigma^s - 4\hat{\sigma}\hat{\rho}\sigma^r\rho^s + 4\hat{\rho}\hat{\sigma}\sigma^{r+s} + 4\hat{\sigma}\hat{\rho}\rho^{r+s})}{(\sigma - \rho)^2} \right] \\ &= \left(\frac{2\hat{\sigma}\sigma^{n+1} - 2\hat{\rho}\rho^{n+1}}{\sigma - \rho} - \frac{2\hat{\sigma}\sigma^{n+r+1} - 2\hat{\rho}\rho^{n+r+1}}{\sigma - \rho} \right) I \\ &\quad + I \left(\frac{2\hat{\sigma}\sigma^{n+r+s+1} - 2\hat{\rho}\rho^{n+r+s+1}}{\sigma - \rho} - \frac{2\hat{\sigma}\sigma^{n+s+1} - 2\hat{\rho}\rho^{n+s+1}}{\sigma - \rho} \right) \\ &\quad + 4(-1)^{n+1} F_r(F_s(CL_0 - 1) + L_s(\mathbf{i} - \mathbf{j} + \mathbf{k})) \\ &= (CLe_n - CLe_{n+r})I + I(CLe_{n+r+s} - CLe_{n+s}) \\ &\quad + 4(-1)^{n+1} F_r(F_s(CL_0 - 1) + L_s(\mathbf{i} - \mathbf{j} + \mathbf{k})). \end{aligned} \quad (62)$$

Thus, Equation (61) is obtained. $\square$

Theorem 7. *For positive integers n and m , where $n \geq m$, the following identities hold true:*

$$CLe_{n+m} + (-1)^m CLe_{n-m} = L_m CLe_n + I(L_m - 1 - (-1)^m), \quad (63)$$

and

$$CL_{n+m} - (-1)^m CL_{n-m} = (Le_{m-1} + 1)CL_{n+1} + I((-1)^m - 1), \quad (64)$$

where L_n denotes the n -th Lucas number, Le_n represents the n -th Leonardo number, and CL_n stands for the n -th Lucas Cartan number.

Proof 7. Applying Binet's formula (35) to the LHS of Equation (63) and substituting Equations (46) and (47) into the Binet representation yield the following identity:

$$\begin{aligned} \text{LHS} &= \frac{2\hat{\sigma}\sigma^{n+m+1} - 2\hat{\rho}\rho^{n+m+1} + 2\hat{\sigma}\sigma^{n-m+1}(-1)^m}{\sigma - \rho} \\ &\quad - \frac{2\hat{\rho}\rho^{n-m+1}(-1)^m}{\sigma - \rho} - I - I(-1)^m \\ &= \frac{2\hat{\sigma}\sigma^m\sigma^{n+1} - 2\hat{\rho}\rho^m\rho^{n+1} + 2\hat{\sigma}\sigma^m\sigma^{n+1} - 2\hat{\rho}\rho^m\rho^{n+1}}{\sigma - \rho} \\ &\quad - I - I(-1)^m \\ &= (\sigma^m + \rho^m) \left(\frac{2\hat{\sigma}\sigma^{n+1} - 2\hat{\rho}\rho^{n+1}}{\sigma - \rho} - I \right) \\ &\quad + I(\sigma^m + \rho^m - (-1)^m - 1) \\ &= L_m CL_n + I(L_m - 1 - (-1)^m). \end{aligned} \quad (65)$$

The approach to proving (64) follows a similar method. $\square$

Theorem 8. For positive integers n and m with $n \geq 1$ and $m \geq 1$, the following identity holds

$$\begin{aligned} CL_{m+1}CL_{n+1} - CL_{m-1}CL_{n-1} &= 4[(31 + CL_2)F_{n+m} \\ &\quad + (14 + CF_2)L_{n+m}] \\ &\quad - (CLE_m I + ICL_n) \\ &\quad - 4(1 + I), \end{aligned} \quad (66)$$

where L_n and Le_n denote the n -th Lucas and Leonardo numbers, respectively.

To illustrate the validity of Equation (66), we consider the case $m = 1$ and $n = 2$.

Using the definition of Leonardo Cartan numbers, we have

$$\begin{aligned} CL_0 &= 1 + \mathbf{i} + 3\mathbf{j} + 5\mathbf{k}, \\ CL_1 &= 1 + 3\mathbf{i} + 5\mathbf{j} + 9\mathbf{k}, \\ CL_2 &= 3 + 5\mathbf{i} + 9\mathbf{j} + 15\mathbf{k}, \\ CL_3 &= 5 + 9\mathbf{i} + 15\mathbf{j} + 25\mathbf{k}. \end{aligned} \quad (67)$$

Then, the LHS of Equation (66) becomes

$$CL_2 CL_3 - CL_0 CL_1 = 454 + 46\mathbf{i} + 80\mathbf{j} + 140\mathbf{k}. \quad (68)$$

On the other hand, using $F_3 = 2$, $L_3 = 4$,

$$\begin{aligned} CF_2 &= 1 + 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}, \\ CL_2 &= 3 + 4\mathbf{i} + 7\mathbf{j} + 11\mathbf{k}, \end{aligned} \quad (69)$$

the right-hand side of Equation (66) is also computed as

$$454 + 46\mathbf{i} + 80\mathbf{j} + 140\mathbf{k}. \quad (70)$$

Thus, both sides coincide, and the identity is verified for $m = 1$ and $n = 2$.

Proof 8. Applying Binet's formula for Leonardo Cartan numbers to the LHS yields

$$\begin{aligned} \text{LHS} &= \left(\frac{2\hat{\sigma}\sigma^{m+2} - 2\hat{\rho}\rho^{m+2}}{\sigma - \rho} - I \right) \left(\frac{2\hat{\sigma}\sigma^{n+2} - 2\hat{\rho}\rho^{n+2}}{\sigma - \rho} - I \right) \\ &\quad - \left(\frac{2\hat{\sigma}\sigma^m - 2\hat{\rho}\rho^m}{\sigma - \rho} - I \right) \left(\frac{2\hat{\sigma}\sigma^n - 2\hat{\rho}\rho^n}{\sigma - \rho} - I \right) \\ &= \frac{4}{(\sigma - \rho)^2} \left(\hat{\sigma}^2 \sigma^{n+m+4} + \hat{\rho}^2 \rho^{n+m+4} - \hat{\sigma}^2 \sigma^{n+m} - \hat{\rho}^2 \rho^{n+m} \right) \\ &\quad + \left(\frac{2\hat{\sigma}\sigma^m - 2\hat{\rho}\rho^m}{\sigma - \rho} - \frac{2\hat{\sigma}\sigma^{m+2} - 2\hat{\rho}\rho^{m+2}}{\sigma - \rho} \right) I \\ &\quad + I \left(\frac{2\hat{\sigma}\sigma^n - 2\hat{\rho}\rho^n}{\sigma - \rho} - \frac{2\hat{\sigma}\sigma^{n+2} - 2\hat{\rho}\rho^{n+2}}{\sigma - \rho} \right) \\ &= 4[(31 + CL_2)F_{n+m} + (14 + CF_2)L_{n+m}] \\ &\quad + (CLE_{m-1} - CLE_{m+1})I + I(CLE_{n-1} - CLE_{n+1}) \\ &= 4[(31 + CL_2)F_{n+m} + (14 + CF_2)L_{n+m}] \\ &\quad - (CLE_m I + ICL_n) - 4(1 + I). \end{aligned} \quad (71)$$

Thus, Equation (66) is obtained. $\square$

Theorem 9. For positive integers m, k and s , where $m \geq k$ and $m \geq s$, the following identity holds

$$\begin{aligned} CL_{m+k}CL_{m-k} - CL_{m+s}CL_{m-s} &= 4(-1)^{m-k-s} (F_{k+s} - L_k F_s) [F_{k+s}(CL_0 - 1) \\ &\quad + L_{k+s}(-\mathbf{i} + \mathbf{j} - \mathbf{k})] + (CLE_{m+s} - CLE_{m+k})I \\ &\quad + I(CLE_{m-s} - CLE_{m-k}), \end{aligned} \quad (72)$$

where F_n is n -th Fibonacci number.

Proof 9. Applying Binet's formula for Leonardo Cartan numbers to the LHS yields

$$\begin{aligned} \text{LHS} &= \frac{4(-1)^{m+1}}{(\sigma - \rho)^2} (\hat{\sigma}\hat{\rho}(\sigma^s \rho^{-s} - \sigma^k \rho^{-k}) + \hat{\rho}\hat{\sigma}(\sigma^{-s} \rho^s - \sigma^{-k} \rho^k)) \\ &\quad + \left(\frac{2\hat{\sigma}\sigma^{m+s+1} - 2\hat{\rho}\rho^{m+s+1}}{\sigma - \rho} - \frac{2\hat{\sigma}\sigma^{m+k+1} - 2\hat{\rho}\rho^{m+k+1}}{\sigma - \rho} \right) I \\ &\quad - I \left(\frac{2\hat{\sigma}\sigma^{m-k+1} - 2\hat{\rho}\rho^{m-k+1}}{\sigma - \rho} - \frac{2\hat{\sigma}\sigma^{m-s+1} - 2\hat{\rho}\rho^{m-s+1}}{\sigma - \rho} \right) \\ &= 4(-1)^{m-k-s} (F_{k+s} - L_k F_s) [F_{k+s}(CL_0 - 1) \\ &\quad + L_{k+s}(-\mathbf{i} + \mathbf{j} - \mathbf{k})] + (CLE_{m+s} - CLE_{m+k})I \\ &\quad + I(CLE_{m-s} - CLE_{m-k}), \end{aligned} \quad (73)$$

where F_n , L_n , and CF_n represent the n -th Fibonacci, Lucas, and Fibonacci Cartan numbers, respectively. $\square$

Theorem 10. Let k, m, s , and t be positive integers such that $k \geq m$ and $s \geq t$. Under the condition $k + m = s + t$, the following identity is obtained:

$$\begin{aligned} CL_{ek}CL_{em} - CL_{es}CL_{et} &= 4[(F_sF_t - F_kF_m)(CL_0 - 1) \\ &\quad - (F_kL_m - F_sL_t)(\mathbf{i} - \mathbf{j} + \mathbf{k})] \\ &\quad + (CL_{es} - CL_{ek})I \\ &\quad + I(CL_{et} - CL_{em}), \end{aligned} \quad (74)$$

where F_n and L_n are n -th Fibonacci and Lucas numbers.

Proof 10. Applying Binet's formula for Leonardo Cartan numbers to the LHS yields

$$\begin{aligned} \text{LHS} &= \frac{1}{(\sigma - \rho)^2} [(2\hat{\sigma}\sigma^{k+1} - 2\hat{\rho}\rho^{k+1} - I(\sigma - \rho))(2\hat{\sigma}\sigma^{m+1} \\ &\quad - 2\hat{\rho}\rho^{m+1} - I(\sigma - \rho)) - (2\hat{\sigma}\sigma^{s+1} - 2\hat{\rho}\rho^{s+1} \\ &\quad - I(\sigma - \rho))(2\hat{\sigma}\sigma^{t+1} - 2\hat{\rho}\rho^{t+1} - I(\sigma - \rho))] \\ &= \frac{4}{(\sigma - \rho)^2} (\hat{\rho}\hat{\sigma}\sigma^m\rho^k + \hat{\sigma}\hat{\rho}\sigma^k\rho^m - \hat{\rho}\hat{\sigma}\sigma^t\rho^s - \hat{\sigma}\hat{\rho}\sigma^s\rho^t) \\ &\quad + \left(\frac{2\hat{\sigma}\sigma^{s+1} - 2\hat{\rho}\rho^{s+1}}{\sigma - \rho} - \frac{2\hat{\sigma}\sigma^{k+1} - 2\hat{\rho}\rho^{k+1}}{\sigma - \rho} \right) I \\ &\quad + I \left(\frac{2\hat{\sigma}\sigma^{t+1} - 2\hat{\rho}\rho^{t+1}}{\sigma - \rho} - \frac{2\hat{\sigma}\sigma^{m+1} - 2\hat{\rho}\rho^{m+1}}{\sigma - \rho} \right) \\ &= 4[(F_sF_t - F_kF_m)(CL_0 - 1) - (F_kL_m - F_sL_t)(\mathbf{i} - \mathbf{j} + \mathbf{k})] \\ &\quad + (CL_{es} - CL_{ek})I + I(CL_{et} - CL_{em}). \end{aligned} \quad (75)$$

Thus, Equation (74) is obtained. $\square$

Theorem 11. For positive integers n and m , with $n \geq m$, the identities are given by

$$\begin{aligned} CF_nCL_{em} + CF_mCL_{en} &= \frac{2}{5}[L_{m+n+1}(23 + 2CL_0) \\ &\quad - (-1)^m L_{n-m}(1 + 6\mathbf{i} - 2\mathbf{j} + 9\mathbf{k})] \\ &\quad + F_{m+n+1}(22 + 4CF_0) \\ &\quad - (CF_n + CF_m)I, \end{aligned} \quad (76)$$

$$\begin{aligned} CF_nCL_{em} - CF_mCL_{en} &= 2(-1)^m F_{n-m}(1 + 2\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}) \\ &\quad - (CF_n - CF_m)I, \end{aligned} \quad (77)$$

where F_n , L_n , CF_n , and CL_n denote the n -th Fibonacci, Lucas, Fibonacci Cartan, and Lucas Cartan numbers, respectively.

Proof 11. To prove Equation (76), Binet's formulas for the Fibonacci Cartan and Leonardo Cartan numbers are applied, yielding the following result:

$$\begin{aligned} \text{LHS} &= \left(\frac{\hat{\sigma}\sigma^n - \hat{\rho}\rho^n}{\sigma - \rho} \right) \left(\frac{2\hat{\sigma}\sigma^{m+1} - 2\hat{\rho}\rho^{m+1} - I(\sigma - \rho)}{\sigma - \rho} \right) \\ &\quad + \left(\frac{\hat{\sigma}\sigma^m - \hat{\rho}\rho^m}{\sigma - \rho} \right) \left(\frac{2\hat{\sigma}\sigma^{n+1} - 2\hat{\rho}\rho^{n+1} - I(\sigma - \rho)}{\sigma - \rho} \right) \\ &= \frac{1}{(\sigma - \rho)^2} [4\hat{\sigma}^2\sigma^{m+n+1} + 4\hat{\rho}^2\rho^{m+n+1} - 2(-1)^m(\hat{\sigma}\hat{\rho}\sigma^{n-m}\rho \\ &\quad + \hat{\rho}\hat{\sigma}\sigma^{n-m} + \hat{\sigma}\hat{\rho}\rho^{n-m+1} + \hat{\rho}\hat{\sigma}\sigma^{n-m+1})] \\ &\quad - \left(\frac{\hat{\sigma}\sigma^n - \hat{\rho}\rho^n}{\sigma - \rho} + \frac{\hat{\sigma}\sigma^m - \hat{\rho}\rho^m}{\sigma - \rho} \right) I \\ &= \frac{2}{5}[L_{m+n+1}(23 + 2CL_0) - (-1)^m L_{n-m}(1 + 6\mathbf{i} - 2\mathbf{j} + 9\mathbf{k})] \\ &\quad + F_{m+n+1}(22 + 4CF_0) - (CF_n + CF_m)I. \end{aligned} \quad (78)$$

The method used to prove equation (77) is analogous to the one applied previously. $\square$

3 | Conclusion

In this study, the fundamental properties of Leonardo Cartan numbers were investigated, with a focus on their recurrence relations, summation formulas, and key mathematical identities such as the Catalan and Cassini identities. By employing Binet's formula, various algebraic properties were derived, and the connections between Fibonacci, Lucas, and Leonardo sequences were demonstrated. The findings contribute to the broader understanding of integer sequences and their extensions.

Although the present work is primarily theoretical, it is worth noting that Cartan numbers are isomorphic to the split quaternion algebra and therefore admit matrix representations. Consequently, Leonardo Cartan numbers can also be expressed in matrix form through this correspondence. However, such representations and their potential applications are beyond the scope of the present study.

Future research may explore further generalizations of Leonardo Cartan numbers and investigate their applications in different mathematical structures.

Author Contributions

All authors contributed equally to this work. They have accepted responsibility for the entire content of the manuscript and approved its submission, results, and final version.

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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