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Split Quaternionic Representations of Horadam Sequences and Their Binet, Generating Function, and Cassini-Type Identities

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ABSTRACT

This study establishes a novel algebraic connection between Horadam numbers and the split quaternion algebra. To this end, two fundamental constructs are introduced: the Fibonacci $\mathbf{S}_{q,r}$-split quaternions and the Horadam $\mathbf{s}_{q,r}$-split quaternions, which generalize Horadam numbers within the framework of split quaternions. Several key results are derived, including the Binet-like formula, the generating function, and the Cassini-type identity, each revealing new structural relationships between these algebraic systems. The findings not only extend the scope of Horadam sequences into the realm of hypercomplex numbers but also open potential applications in number theory and algebraic analysis. İskender Öztürk and Hasan Çakır contributed equally to this study.

1 | Introduction

In recent years, hypercomplex number systems have attracted considerable attention due to their wide-ranging applications in geometry, physics, and algebraic structures. Among these, the split quaternions (also known as coquaternions), introduced by James Cockle in 1849, form an associative algebra over the real numbers. Their algebraic and geometric structures have been employed in many contexts. For instance, Kula and Yaylı [1] investigated their role in Lorentzian geometry, Özdemir and Ergin [2] studied their geometric interpretations in Minkowski space, and recently, Öztürk and Özdemir [3] analyzed their connections with spacelike and timelike frames. Beyond geometry, the algebraic nature of split quaternions has also been utilized in matrix theory, as demonstrated by Erdoğan and Özdemir [4], Ni et al. [5], Si et al. [6], and Wang et al. [7]. Furthermore, their applications in mechanics have been explored in several works, including Brody and Graefe [8], Jiang et al. [9], and others [10–12]. This breadth of applications highlights the versatility of split quaternions in both pure and applied mathematics.

The structural richness of split quaternions has also inspired researchers to extend various number sequences into the split quaternionic framework. In particular, quaternionic analogs of the Fibonacci and Lucas sequences were introduced and studied in different contexts [13–16], while Akyiğit et al. [17] formulated more general Horadam-type constructions. A comprehensive treatment of these extensions and their algebraic implications can also be found in Öztürk and Çakır's study [18]. Consequently, split quaternions have emerged as an important tool in number theory as well as in geometry.

In addition to classical quaternionic constructions, several recent studies have investigated quaternionic sequences associated with various number families and hypercomplex structures. In particular, quaternionic extensions involving Leonardo numbers, repunit numbers, and related algebraic systems have been examined in different contexts [19–23]. These studies highlight the increasing interest in extending recurrence sequences into quaternionic and hybrid algebraic frameworks and provide further motivation for exploring Horadam-type sequences within the split quaternion algebra.

The Horadam numbers, introduced by Horadam et al. in the 1960s, represent a generalization of several well-known second-order recurrence sequences, including the Fibonacci, Lucas, and Pell numbers. Defined by four parameters, these sequences possess remarkable flexibility and have been extensively studied for their algebraic and combinatorial properties [24–29]. Their matrix representations and determinant properties have been analyzed in various studies, such as those by Bueno [30], Cerdá-Morales [31], Chen and Chen [32], Shi [33], and Melham and Shannon [34]. Later works by Wani and Koul [35, 36] further developed their matrix forms and generating functions. In recent years, Horadam-type sequences have also been applied to cryptographic schemes due to their recursive and modular properties [37–44], reinforcing their relevance in both theoretical and applied domains. In addition, Horadam-type sequences and related Fibonacci-based structures have also been studied in various numerical and computational contexts. For example, Horadam and Fibonacci polynomials have been used in collocation-based numerical approaches for solving nonlinear and fractional differential equations [45–48].

Despite the extensive studies on both Horadam sequences and split quaternions, a direct algebraic connection between these two structures has not yet been systematically explored. Establishing such a relationship is significant because it extends number-theoretic recursions into the hypercomplex domain, allowing for new interpretations of recurrence relations, matrix representations, and generating functions in a broader algebraic context. This motivates the present study, which aims to reveal how Horadam numbers can be characterized and generalized within the split quaternion algebra.

The aim of this paper is to establish a new algebraic relationship between Horadam numbers and split quaternions. By exploiting the isomorphism between split quaternions and 2×2 real matrices, we construct the split quaternion that generates Horadam numbers and introduce two fundamental quaternionic extensions: the Fibonacci $\mathbf{S}_{q,r}$-split quaternions and the Horadam $\mathbf{s}_{q,r}$-split quaternions.

Based on this construction, several key results are derived, including Binet-like expressions, generating functions, and Cassini-type identities. Through these findings, the study provides an explicit algebraic bridge between Horadam sequences and the split quaternion algebra. Consequently, the results enrich both number theory and hypercomplex algebra, offering a unified framework that may inspire further research in algebraic geometry, combinatorial analysis, and applied mathematics.

The paper is organized as follows. Section 2 presents the necessary preliminaries by recalling the fundamental properties of split quaternions and Horadam numbers that constitute the algebraic framework of the study. Section 3 introduces the Fibonacci $\mathbf{S}_{q,r}$-split quaternions and derives their main characteristics, including the Binet formula and several related identities. Section 4 develops the concept of Horadam $\mathbf{s}_{q,r}$-split quaternions and establishes further results such as the generating function and Cassini-type identity. Finally, Section 5 concludes the paper with a concise discussion of the obtained results and outlines possible directions for future research.

2 | Preliminaries

In this section, we recall the basic algebraic properties and definitions that will be used throughout the paper. For completeness, we first summarize the fundamental structure of split quaternions and then review the main characteristics of Horadam numbers, which together form the algebraic foundation of the subsequent sections.

2.1 | Split Quaternions

The set of split quaternions can be represented as

$$\widehat{\mathbb{H}} = \left\{ a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} : \mathbf{i}^2 = -1, \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = 1 \right\}, \tag{1}$$

where a, b, c, d are real numbers [1, 2]. Split quaternions are noncommutative under multiplication, but they are associative. The set of split quaternions also has zero divisors, nontrivial idempotent, and nilpotent elements [1, 2]. A split quaternion $\mathbf{s} = s_1 + s_2\mathbf{i} + s_3\mathbf{j} + s_4\mathbf{k}$ can be represented in the form $\mathbf{s} = S_\mathbf{s} + \mathbf{V_s}$, where $S_\mathbf{s} = s_1$ represents the scalar part of $\mathbf{s}$ and $\mathbf{V_s} = s_2\mathbf{i} + s_3\mathbf{j} + s_4\mathbf{k}$ represents the vector part of $\mathbf{s}$. If $S_\mathbf{s} = 0$, then $\mathbf{s}$ is said to be a pure split quaternion. The conjugate of the split quaternion $\mathbf{s} = s_1 + s_2\mathbf{i} + s_3\mathbf{j} + s_4\mathbf{k}$ is indicated by $\bar{\mathbf{s}}$, and that is $\bar{\mathbf{s}} = S_\mathbf{s} - \mathbf{V_s} = s_1 - s_2\mathbf{i} - s_3\mathbf{j} - s_4\mathbf{k}$.

For any $\mathbf{s}, \mathbf{w} \in \widehat{\mathbb{H}}$, the product of any two split quaternions $\mathbf{s}$ and $\mathbf{w}$ is $\mathbf{sw} = (S_\mathbf{s}S_\mathbf{w} + \langle \mathbf{V_s}, \mathbf{V_w} \rangle_{\mathbb{L}}) + (S_\mathbf{s}\mathbf{V_w} + S_\mathbf{w}\mathbf{V_s} + \mathbf{V_s} \times_{\mathbb{L}} \mathbf{V_w})$, respectively, where $\langle , \rangle_{\mathbb{L}}$ represents the Lorentzian scalar product and $\times_{\mathbb{L}}$ represents Lorentzian vector product. The norm of any split quaternion $\mathbf{s}$ is expressed as $N(\mathbf{s}) = \sqrt{|\mathbf{s}\bar{\mathbf{s}}|} = \sqrt{|s_1^2 + s_2^2 - s_3^2 - s_4^2|}$. If $N(\mathbf{s}) = 1$, then $\mathbf{s}$ is said to be a unit split quaternion. The character of a split quaternion is stated by $I(\mathbf{s}) = \mathbf{s}\bar{\mathbf{s}} = \bar{\mathbf{s}}\mathbf{s} = s_1^2 + s_2^2 - s_3^2 - s_4^2$. Any split quaternion is said to be spacelike, timelike, or lightlike if $I(\mathbf{s}) < 0$, $I(\mathbf{s}) > 0$, or $I(\mathbf{s}) = 0$, respectively. A pure split quaternion corresponds to a vector in $\mathbb{E}_1^3$. The character of a pure split quaternion is expressed as the product of $I(\mathbf{V_s}) = -s_2^2 + s_3^2 + s_4^2$. A pure split quaternion is spacelike, timelike, or lightlike if $I(\mathbf{V_s}) > 0$, $I(\mathbf{V_s}) < 0$, or $I(\mathbf{V_s}) = 0$, respectively [1, 2, 4, 49, 50]. In [2], the polar form of any split quaternion is stated as below:

i. If $\mathbf{s}$ is a timelike split quaternion and its vector part is spacelike, then it can be stated as

$$\mathbf{s} = N(\mathbf{s})(\cosh\theta + \mathbf{V_s}\sinh\theta), \tag{2}$$

where $\mathbf{V_s}$ is a spacelike unit vector and $\cosh\theta = (|s_1|)/N(\mathbf{s})$, $\sinh\theta = \sqrt{I(\mathbf{V_s})}/N(\mathbf{s})$, $\mathbf{V_s} = (s_2\mathbf{i} + s_3\mathbf{j} + s_4\mathbf{k})/\sqrt{I(\mathbf{V_s})}$, and $\mathbf{V_s}^2 = 1$.

ii. If $\mathbf{s}$ is a timelike split quaternion and its vector part is timelike, then it can be stated as

$$\mathbf{s} = N(\mathbf{s})(\cos\theta + \mathbf{V_s}\sin\theta), \tag{3}$$

where $\mathbf{V_s}$ is a timelike unit vector, and $\cos\theta = s_1/N(\mathbf{s})$, $\sin\theta = \sqrt{-I(\mathbf{V_s})}/N(\mathbf{s})$, $\mathbf{V_s} = (s_2\mathbf{i} + s_3\mathbf{j} + s_4\mathbf{k})/\sqrt{-I(\mathbf{V_s})}$, and $\mathbf{V_s}^2 = -1$.

iii. If $\mathbf{s}$ is a spacelike split quaternion, then it can be stated as

$$\mathbf{s} = N(\mathbf{s})(\sinh\theta + \mathbf{V_s}\cosh\theta), \tag{4}$$

where $\mathbf{V_s}$ is a spacelike unit vector, and $\sinh\theta = s_1/N(\mathbf{s})$, $\cosh\theta = \sqrt{I(\mathbf{V_s})}/N(\mathbf{s})$, $\mathbf{V_s} = (s_2\mathbf{i} + s_3\mathbf{j} + s_4\mathbf{k})/\sqrt{I(\mathbf{V_s})}$, and $\mathbf{V_s}^2 = 1$.

For any split quaternion, De Moivre formulas are given in [50]. Here, for $n \in \mathbb{N}$, we give these formulas as below:

i. Let $\mathbf{s} = N(\mathbf{s})(\cosh\theta + \mathbf{V_s}\sinh\theta)$ be a timelike split quaternion and its vector part is a spacelike vector then

$$\mathbf{s}^n = (N(\mathbf{s}))^n(\cosh n\theta + \mathbf{V_s}\sinh n\theta), \tag{5}$$

ii. Let $\mathbf{s} = N(\mathbf{s})(\cos\theta + \mathbf{V_s}\sin\theta)$ be a timelike split quaternion and its vector part is a timelike vector, then

$$\mathbf{s}^n = (N(\mathbf{s}))^n(\cos n\theta + \mathbf{V_s}\sin n\theta), \tag{6}$$

iii. Let $\mathbf{s} = N(\mathbf{s})(\sinh\theta + \mathbf{V_s}\cosh\theta)$ be a spacelike split quaternion, then

$$\begin{aligned} \mathbf{s}^n &= (N(\mathbf{s}))^n(\cosh n\theta + \mathbf{V_s}\sinh n\theta), \quad \text{if } n \text{ is even},\\ \mathbf{s}^n &= (N(\mathbf{s}))^n(\sinh n\theta + \mathbf{V_s}\cosh n\theta), \quad \text{if } n \text{ is odd}. \end{aligned} \tag{7}$$

In [4, 5, 7, 10–12], the split quaternion ring $\widehat{\mathbb{H}}$ is isomorphic to the ring of real 2×2 matrices. That is the transformation that maps units $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ to 2×2 real matrices as follows:

$$1 \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{i} \longrightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \mathbf{j} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \mathbf{k} \longrightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{8}$$

is a ring isomorphism. The 2×2 real matrix representation of a split quaternion $\mathbf{a} = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ is

$$\begin{bmatrix} a+c & -b+d \\ b+d & a-c \end{bmatrix}. \tag{9}$$

Conversely, the split quaternion representation of a 2×2 real matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\frac{a+d}{2} + \frac{c-b}{2}\mathbf{i} + \frac{a-d}{2}\mathbf{j} + \frac{c+b}{2}\mathbf{k}. \tag{10}$$

For detailed information about the real matrix representation of split quaternions and related hypercomplex matrix structures, refer to [1, 4, 5, 7, 10–12].

Recent studies have also explored various quaternionic and matrix-based structures associated with number sequences and hypercomplex systems, further emphasizing the interaction between matrix methods and generalized quaternionic models [51–54].

2.2 | Horadam Numbers

The number sequence $w_n = w_n(a, b;\, q, r)$ is defined by a recursive relation

$$w_n = qw_{n-1} - rw_{n-2}, \tag{11}$$

for $n > 1$ with initial conditions $w_0 = a, w_1 = b$. Then, the Binet formula for the Horadam number

$$w_n = \alpha_1 x_1^n + \alpha_2 x_2^n, \tag{12}$$

where $\alpha_1 = (b - ax_2)/(x_1 - x_2)$ and $\alpha_2 = (ax_1 - b)/(x_1 - x_2)$, and x_1, x_2 are roots of the equation $x^2 - qx + r = 0$. In particular, if the parameter q, r is taken $1, -1$, respectively, then the Fibonacci numbers

$$f_n = f_{n-1} + f_{n-2} \text{ with } f_0 = 0, f_1 = 1, \tag{13}$$

which are a special case of w_n, are obtained. Similarly, the Lucas numbers are defined by

$$g_n = g_{n-1} + g_{n-2}, \tag{14}$$

with $g_0 = 2, g_1 = 1$ [26, 55]. In [26], Cassini identity is expressed as follows. Let w_n be $n-$th Horadam sequence of numbers. For $n \geq 1$, Cassini-like identity for w_n is as follows:

$$(w_{n-1})(w_{n+1}) - (w_n)^2 = -r^{n-1}\rho^2, \tag{15}$$

where $\rho^2 = b^2 + ra^2 - qab$.

The generalized Fibonacci–Lucas matrix $M_{q,r} = \begin{bmatrix} q & -r \\ 1 & 0 \end{bmatrix}$ is called generating matrix for the Fibonacci and Lucas numbers. Since this matrix transforms the Fibonacci vector $(f_n, f_{n-1})^T$ whose components are consecutive Fibonacci numbers to the Fibonacci vector $(f_{n+1}, f_n) = M_{q,r}(f_n, f_{n-1})^T$. A similar situation can be given for a Lucas vector $(g_n, g_{n-1})^T$ whose components are consecutive Lucas numbers that is $(g_{n+1}, g_n) = M_{q,r}(g_n, g_{n-1})^T$. Also, for n is a positive integer

$$(M_{q,r})^n = \begin{bmatrix} f_{n+1} & -rf_n \\ f_n & -rf_{n-1} \end{bmatrix}. \tag{16}$$

A detailed explanation of the subject can be found in [30–36].

Some properties of Horadam numbers can be studied with their 2×2 real matrix representations. Similarly, a split quaternion can be represented with 2×2 real matrices. Due to this similarity, split quaternions are used in this article to produce Horadam numbers and study their various properties. In this respect, it is aimed to provide a different expansion in the area of use of Horadam numbers and split quaternions. For this purpose, a split quaternion is obtained corresponding to a generalized Fibonacci–Lucas matrix. Thus, the multiplication operation of split quaternions, polar representation, and De Moivre formulas are used to derive Horadam numbers and some of their properties.

3 | The Fibonacci $S_{q,r}$-Split Quaternions

Since the ring of split quaternions is isomorphic to the ring of 2×2 real matrices $\mathbb{M}_{2\times 2}$, the split quaternion corresponding to the matrix

$$M_{q,r} = \begin{bmatrix} q & -r \\ 1 & 0 \end{bmatrix}, \tag{17}$$

is $\mathbf{S}^1_{q,r} = (q/2) + ((1+r)/2)\mathbf{i} + (q/2)\mathbf{j} + ((1-r)/2)\mathbf{k} = (q/2) + \mathbf{v}^1_{q,r}$. Thus, the following definition can be given.

This construction is based on the correspondence between split quaternions and their 2×2 real matrix representations. In particular, the generating matrix $M_{q,r}$ induces a quaternionic structure through the standard matrix representation of split quaternions [1, 4, 5].

Definition 1. The Fibonacci $\mathbf{S}_{q,r}$–split quaternions are the sequence of numbers $\left\{\mathbf{S}^n_{q,r}\right\}_{n=1}^{\infty}$ defined by the linear recurrence equation

$$\mathbf{S}^n_{q,r} = q\mathbf{S}^{n-1}_{q,r} - r\mathbf{S}^{n-2}_{q,r}, \quad n > 1, \tag{18}$$

with initial conditions $\mathbf{S}^0_{q,r} = 1$ and $\mathbf{S}^1_{q,r} = (q/2) + ((1+r)/2)\mathbf{i} + (q/2)\mathbf{j} + ((1-r)/2)\mathbf{k}$ are split quaternions.

Since $N(\mathbf{S}^1_{q,r}) = r$, the characteristic of $\mathbf{S}^1_{q,r}$ is given as follows:

- If $N(\mathbf{S}^1_{q,r}) = r < 0$, then $\mathbf{S}^1_{q,r}$ is spacelike. Thus, its polar form is

$$\mathbf{S}^1_{q,r} = \sqrt{|r|}(\sinh\theta + \cosh\theta\mathbf{x}), \tag{19}$$

where $\sinh\theta = q/(2\sqrt{|r|})$, $\cosh\theta = \sqrt{q^2-4r}/(2\sqrt{|r|})$ and $\mathbf{x} = (2\mathbf{v}^1_{q,r})/\sqrt{q^2-4r}$.

- If $N(\mathbf{S}^1_{q,r}) = r > 0$, then $\mathbf{S}^1_{q,r}$ is timelike. Therefore, the following three different polar representations of the timelike Fibonacci $\mathbf{S}_{q,r}$-split quaternion occur, depending on the character of the vector part.

i. If $q^2/4 - r < 0$, then

$$\mathbf{S}^1_{q,r} = \sqrt{r}(\cos\theta + \sin\theta\mathbf{x}), \tag{20}$$

where $\cos\theta = q/(2\sqrt{r})$, $\sin\theta = \sqrt{4r-q^2}/(2\sqrt{r})$, and $\mathbf{x} = (2\mathbf{v}^1_{q,r})/\sqrt{4r-q^2}$.

ii. If $q^2/4 - r = 0$, then $\mathbf{S}^1_{q,r} = |(q/2)|(\pm 1 + \mathbf{x})$, where $\mathbf{x} = (2\mathbf{v}^1_{q,r})/(|q|)$.

iii. If $q^2/4 - r > 0$, then

$$\mathbf{S}^1_{q,r} = \sqrt{r}(\pm\cosh\theta + \sinh\theta\mathbf{x}), \tag{21}$$

where $\cosh\theta = (|q|)/(2\sqrt{r})$, $\sinh\theta = \sqrt{q^2-4r}/(2\sqrt{r})$, and $\mathbf{x} = (2\mathbf{v}^1_{q,r})/\sqrt{q^2-4r}$.

The sequence $\{u_n\}_{n=0}^{\infty}$ considered below corresponds to a special case of the Horadam sequence obtained by choosing specific initial parameters in the general recurrence relation defined in (11).

Theorem 1. *Let $\mathbf{S}^1_{q,r} = (q/2) + ((1+r)/2)\boldsymbol{i} + (q/2)\boldsymbol{j} + ((1-r)/2)\boldsymbol{k}$ be a split quaternion. Then, $n \geq 1$*

$$\mathbf{S}^n_{q,r} = \frac{u_{n+1} - ru_{n-1}}{2} + \frac{(1+r)u_n}{2}\mathbf{i} + \frac{u_{n+1} + ru_{n-1}}{2}\mathbf{j} + \frac{(1-r)u_n}{2}\mathbf{k}, \tag{22}$$

$$\mathbf{S}^n_{q,r} = \left(\mathbf{S}^1_{q,r}\right)^n, \tag{23}$$

where $\{u_n\}_{n=0}^{\infty}$ is a generalized Fibonacci sequence with recurrence relation $u_n = qu_{n-1} - ru_{n-2}$ and initial condition $u_0 = 0$, $u_1 = 1$.

Proof 1. We prove Equation (22). We prove it by mathematical induction on n. Demonstrate that the assertion is true for $n = 1$. Thus,

$$\begin{aligned} \mathbf{S}^1_{q,r} &= \frac{u_2 - ru_0}{2} + \frac{(1+r)u_1}{2}\mathbf{i} + \frac{u_2 + ru_0}{2}\mathbf{j} + \frac{(1-r)u_1}{2}\mathbf{k} \\ &= \frac{q}{2} + \frac{1+r}{2}\mathbf{i} + \frac{q}{2}\mathbf{j} + \frac{1-r}{2}\mathbf{k}. \end{aligned} \tag{24}$$

Hence, Equation (22) holds when $n = 1$. Assume Equation (22) holds when $n = k$ for any integer $k \geq 1$; that is, assume that

$$\mathbf{S}^k_{q,r} = \frac{u_{k+1} - ru_{k-1}}{2} + \frac{(1+r)u_k}{2}\mathbf{i} + \frac{u_{k+1} + ru_{k-1}}{2}\mathbf{j} + \frac{(1-r)u_k}{2}\mathbf{k}. \tag{25}$$

We want to prove that it remains valid when $n = k + 1$. By the inductive hypothesis, we obtain

$$\begin{aligned} \mathbf{S}^{k+1}_{q,r} &= q\mathbf{S}^k_{q,r} - r\mathbf{S}^{k-1}_{q,r} \\ &= q\left(\frac{u_{k+1} - ru_{k-1}}{2} + \frac{(1+r)u_k}{2}\mathbf{i} + \frac{u_{k+1} + ru_{k-1}}{2}\mathbf{j} + \frac{(1-r)u_k}{2}\mathbf{k}\right) \\ &\quad - r\left(\frac{u_k - ru_{k-2}}{2} + \frac{(1+r)u_{k-1}}{2}\mathbf{i} + \frac{u_k + ru_{k-2}}{2}\mathbf{j} + \frac{(1-r)u_{k-1}}{2}\mathbf{k}\right) \\ &= \frac{qu_{k+1} - ru_k - r(qu_{k-1} - ru_{k-2})}{2} + \frac{(1+r)(qu_k - ru_{k-1})}{2}\mathbf{i} \\ &\quad + \frac{qu_{k+1} - ru_k + r(qu_{k-1} - ru_{k-2})}{2}\mathbf{j} + \frac{(1-r)(qu_k - ru_{k-1})}{2}\mathbf{k} \\ &= \frac{u_{k+2} - ru_k}{2} + \frac{(r+1)u_{k+1}}{2}\mathbf{i} + \frac{u_{k+2} + ru_k}{2}\mathbf{j} + \frac{(1-r)u_{k+1}}{2}\mathbf{k}. \end{aligned} \tag{26}$$

We have thus shown that if Equation (22) holds for $n = k$, it must also hold for $n = k + 1$. By the principle of mathematical induction, the result follows for all positive integers n. Equation (23) can be proven in a similar way. □

Corollary 1. *If $r > 0$, then the polar form of $\mathbf{S}^n_{q,r}$ is as follows.*

$$\begin{aligned} \mathbf{S}^n_{q,r} &= \sqrt{r}^n(\cos n\theta + \sin n\theta\mathbf{x})\,\text{and}\,\mathbf{x} = \frac{2\mathbf{v}^1_{q,r}}{\sqrt{|q^2-4r|}}, \quad \text{if}\,\frac{q^2}{4} - r < 0, \\ \mathbf{S}^n_{q,r} &= \left|\frac{q}{2}\right|^n(\pm 1 + n\mathbf{x})\,\text{and}\,\mathbf{x} = \frac{2\mathbf{v}^1_{q,r}}{|q|}, \quad \text{if}\,\frac{q^2}{4} - r = 0, \\ \mathbf{S}^n_{q,r} &= \sqrt{r}^n(\pm\cosh n\theta + \sinh n\theta\mathbf{x})\,\text{and}\,\mathbf{x} = \frac{2\mathbf{v}^1_{q,r}}{\sqrt{q^2-4r}}, \quad \text{if}\,\frac{q^2}{4} - r > 0, \end{aligned} \tag{27}$$

where $\cos\theta = (q/2)/r$ and $\cosh\theta = (|q/2|)/r$.

Proof 2. This can be seen clearly from the De Moivre formulas for split quaternions. □

For example, for $r > 0$, the polar form of the Fibonacci $\mathbf{S}_{q,r}$-split quaternion $\mathbf{S}_{q,r}\mathbf{S}^n_{q,r}$ with respect to the parameters q, r of the equation $x^2 - qx + r = 0$ is shown in Table 1, where $\mathbf{u} = (3\mathbf{i} + 2\mathbf{j} - \mathbf{k})/2$, $\mathbf{w} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Corollary 2. *If $r < 0$, then the polar form of $\mathbf{S}^n_{q,r}$ is as follows:*

$$\begin{aligned} \mathbf{S}^n_{q,r} &= \sqrt{|r|}^n(\sinh n\theta + \cosh n\theta\mathbf{x}), \quad \text{if}\, n\, \text{is odd}, \\ \mathbf{S}^n_{q,r} &= \sqrt{|r|}^n(\cosh n\theta + \sinh\theta\mathbf{x}), \quad \text{if}\, n\, \text{is even}, \end{aligned} \tag{28}$$

where $\sinh\theta = q/(2\sqrt{|r|})$, $\cosh\theta = \sqrt{q^2-4r}/(2\sqrt{|r|})$, *and* $\mathbf{x} = (2\mathbf{v}^1_{q,r})/\sqrt{q^2-4r}$.

Proof 3. This can be seen clearly from the De Moivre formulas for split quaternions. □

For example, for $r < 0$, the polar form of the Fibonacci $\mathbf{S}_{q,r}$-split quaternion $\mathbf{S}^n_{q,r}$ according to the parameters q, r of the equation $x^2 - qx + r = 0$ is exhibited in Table 2.

Corollary 3. *Let n be a positive integer. Then, the following statement is valid:*

i. $N(\mathbf{S}^n_{q,r}) = r^n$

ii. $\mathbf{i}\mathbf{S}^n_{q,r} + \overline{\mathbf{i}\mathbf{S}^n_{q,r}} = -(1+r)u_n$, $\mathbf{k}\mathbf{S}^n_{q,r} + \overline{\mathbf{k}\mathbf{S}^n_{q,r}} = (1-r)u_n$

Proof 4.

i. We prove it by direct calculation.

$$\begin{aligned} N\left(\mathbf{S}^n_{q,r}\right) &= \left(\frac{u_{n+1} - ru_{n-1}}{2}\right)^2 + \left(\frac{(1+r)u_n}{2}\right)^2 \\ &\quad - \left(\frac{u_{n+1} + ru_{n-1}}{2}\right)^2 - \left(\frac{(1-r)u_n}{2}\right)^2 \\ &= r\left(u_n^2 - u_{n-1}u_{n+1}\right) = r\left(r^{n-1}\right) = r^n. \end{aligned} \tag{29}$$

ii. This identity follows directly from the multiplication rules of split quaternions. By computing the products $\mathbf{i}\mathbf{S}^n_{q,r}$ and $\overline{\mathbf{i}\mathbf{S}^n_{q,r}}$ using the definition of $\mathbf{S}^n_{q,r}$ in (22) and collecting the scalar and vector parts, the relation $\mathbf{i}\mathbf{S}^n_{q,r} + \overline{\mathbf{i}\mathbf{S}^n_{q,r}} = -(1+r)u_n$ is obtained. A similar calculation gives $\mathbf{k}\mathbf{S}^n_{q,r} + \overline{\mathbf{k}\mathbf{S}^n_{q,r}} = (1-r)u_n$. □

Theorem 2 (Binet-like Formula). *Binet formula for Fibonacci $\mathbf{S}_{q,r}$–split quaternions is*

$$\mathbf{S}^n_{q,r} = \alpha_1\widehat{\alpha_1}x_1^n + \alpha_2\widehat{\alpha_2}x_2^n, \tag{30}$$

where $\alpha_1 = (b - ax_2)/(x_1 - x_2)$, $\alpha_2 = (ax_1 - b)/(x_1 - x_2)$, x_1, x_2 are roots of the equation $x^2 - qx + r = 0$ and $\widehat{\alpha_1} = ((x_1^2 - r)/(2x_1)) + ((1+r)/2)\mathbf{i} + (x_1^2 + r)/(2x_1)\mathbf{j} + ((1-r)/2)\mathbf{k}$, $\widehat{\alpha_2} = ((x_2^2 - r)/(2x_2)) + ((1+r)/2)\mathbf{i} + (x_2^2 + r)/(2x_2)\mathbf{j} + ((1-r)/2)\mathbf{k}$.

Proof 5. We write $\alpha_1x_1^{n-1} + \alpha_2x_2^{n-1}$, $\alpha_1x_1^n + \alpha_2x_2^n$, and $\alpha_1x_1^{n+1} + \alpha_2x_2^{n+1}$ instead of u_{n-1}, u_n, and u_{n+1} in Equation (22), respectively. When the necessary arrangements are made, it can be seen that

TABLE 1 | Polar and numerical representations of $\mathbf{S}^5_{q,r}$ for selected (q, r) pairs[a].

q	r	Polar form of $\mathbf{S}^5_{q,r}$	Numerical value of $\mathbf{S}^5_{q,r}$
2	2	$\sqrt{2}^5(\cos(5\cos^{-1}1/\sqrt{2}) + \sin(5\cos^{-1}1/\sqrt{2})\mathbf{u}_b)$	$-4 - 4\mathbf{u}$
2	1	$\lvert(2/2)\rvert^5(1 + 5(\mathbf{i} + \mathbf{j}))$	$1 + 5\mathbf{i} + 5\mathbf{j}$
−4	3	$\sqrt{3}^5(\cosh(5\cosh^{-1}2/\sqrt{3}) + \sinh(5\cosh^{-1}2/\sqrt{3})\mathbf{w}_b)$	$122 + 121\mathbf{w}$

[a]The polar form corresponds to the timelike representation of $\mathbf{S}^5_{q,r}$ in the split quaternion algebra.
[b]The variable $\mathbf{u}$ (or $\mathbf{w}$) denotes the unit direction in the respective split quaternion form.

TABLE 2 | Polar and numerical forms of $\mathbf{S}_{q,r}^5$ for different (q,r) values[a].

q	r	Polar form of $\mathbf{S}_{q,r}^5 b$	Numerical value of $\mathbf{S}_{q,r}^5$
1	-1	$\sinh(5\sinh^{-1}1/2)+\cosh(5\sinh^{-1}1/2)(\mathbf{j}+2\mathbf{k})/\sqrt{5}$	$11/2+(5/2)\mathbf{j}+5\mathbf{k}$
2	-1	$\sinh(5\sinh^{-1}1)+\cosh(5\sinh^{-1}1)(\mathbf{j}+\mathbf{k})/\sqrt{2}$	$41+29(\mathbf{j}+\mathbf{k})$

[a]The values correspond to the spacelike representation of $\mathbf{S}_{q,r}^5$ in the split quaternion algebra.
[b]Computations are based on the respective (q,r) parameters in the split quaternion algebra.

$$
\begin{aligned}
\mathbf{S}_{q,r}^n &= \frac{u_{n+1}-ru_{n-1}}{2}+\frac{(1+r)u_n}{2}\mathbf{i}+\frac{u_{n+1}+ru_{n-1}}{2}\mathbf{j}+\frac{(1-r)u_n}{2}\mathbf{k}\\
&= \frac{\alpha_1x_1^{n+1}+\alpha_2x_2^{n+1}-r\left(\alpha_1x_1^{n-1}+\alpha_2x_2^{n-1}\right)}{2}+\frac{(1+r)\left(\alpha_1x_1^n+\alpha_2x_2^n\right)}{2}\mathbf{i}+\\
&\quad+\frac{\alpha_1x_1^{n+1}+\alpha_2x_2^{n+1}+r\left(\alpha_1x_1^{n-1}+\alpha_2x_2^{n-1}\right)}{2}\mathbf{j}+\frac{(1-r)\left(\alpha_1x_1^n+\alpha_2x_2^n\right)}{2}\mathbf{k}\\
&= \frac{\alpha_1x_1^{n+1}-r\alpha_1x_1^{n-1}}{2}+\frac{(1+r)\left(\alpha_1x_1^n\right)}{2}\mathbf{i}+\frac{\alpha_1x_1^{n+1}+r\alpha_1x_1^{n-1}}{2}\mathbf{j}+\frac{(1-r)\left(\alpha_1x_1^n\right)}{2}\mathbf{k}\\
&\quad+\frac{\alpha_2x_2^{n+1}-r\alpha_2x_2^{n-1}}{2}+\frac{(1+r)\left(\alpha_2x_2^n\right)}{2}\mathbf{i}+\frac{\alpha_2x_2^{n+1}+r\alpha_2x_2^{n-1}}{2}\mathbf{j}+\frac{(1-r)\left(\alpha_2x_2^n\right)}{2}\mathbf{k}\\
&= \alpha_1x_1^n\left(\frac{x_1-rx_1^{-1}}{2}+\frac{(1+r)}{2}\mathbf{i}+\frac{x_1+rx_1^{-1}}{2}\mathbf{j}+\frac{(1-r)}{2}\mathbf{k}\right)\\
&\quad+\alpha_2x_2^n\left(\frac{x_2-rx_2^{-1}}{2}+\frac{(1+r)}{2}\mathbf{i}+\frac{x_2+rx_2^{-1}}{2}\mathbf{j}+\frac{(1-r)}{2}\mathbf{k}\right).
\end{aligned}
\tag{31}
$$

Thus, Equation (30) is obtained. □

Theorem 3. *Generating function of Fibonacci $\mathbf{S}_{q,r}$–split quaternions is*

$$C_{q,r}(x)=\frac{1-\overline{\mathbf{S}_{q,r}^1}x}{1-qx+rx^2}, \tag{32}$$

where $\mathbf{S}_{q,r}^0=1$, $\mathbf{S}_{q,r}^1=(q/2)+((1+r)/2)\mathbf{i}+(q/2)\mathbf{j}+((1-r)/2)\mathbf{k}$ *is the initial condition.*

Proof 6. Let the function

$$C_{q,r}(x)=\sum_{n=0}^{\infty}\mathbf{S}_{q,r}^n x^n, \tag{33}$$

be the generating function of $\mathbf{S}_{q,r}^n$. Multiplying both sides of (33) by qx and $-rx^2$, respectively, yields the following three equations:

$$
\begin{aligned}
C_{q,r}(x)&=\mathbf{S}_{q,r}^0+\mathbf{S}_{q,r}^1x+\mathbf{S}_{q,r}^2x^2+\mathbf{S}_{q,r}^3x^3+\cdots\\
-qxC_{q,r}(x)&=qx\mathbf{S}_{q,r}^0-q\mathbf{S}_{q,r}^1x^2-q\mathbf{S}_{q,r}^2x^3-q\mathbf{S}_{q,r}^3x^4-\cdots\\
rx^2C_{q,r}(x)&=r\mathbf{S}_{q,r}^0x^2+r\mathbf{S}_{q,r}^1x^3+r\mathbf{S}_{q,r}^2x^4+r\mathbf{S}_{q,r}^3x^5+\cdots.
\end{aligned}
\tag{34}
$$

These equations can be arranged as follows, so

$$
\begin{aligned}
C_{q,r}(x)&=\frac{\sum_{n=0}^{\infty}\mathbf{S}_{q,r}^n x^n-\left(q\sum_{n=0}^{\infty}\mathbf{S}_{q,r}^n x^{n+1}-r\sum_{n=0}^{\infty}\mathbf{S}_{q,r}^n x^{n+2}\right)}{1-qx+rx^2}\\
&=\frac{\mathbf{S}_{q,r}^0+\left(\mathbf{S}_{q,r}^1-q\mathbf{S}_{q,r}^0\right)x+\sum_{n=2}^{\infty}\left(\mathbf{S}_{q,r}^n-\left(q\mathbf{S}_{q,r}^{n-1}-r\mathbf{S}_{q,r}^{n-2}\right)\right)x^n}{1-qx+rx^2},
\end{aligned}
\tag{35}
$$

is obtained. Using the equation $\sum_{n=2}^{\infty}(\mathbf{S}_{q,r}^n-(q\mathbf{S}_{q,r}^{n-1}-r\mathbf{S}_{q,r}^{n-2}))x^n=0$, it can be seen that $C_{q,r}(x)=(1+(\mathbf{S}_{q,r}^1-q)x)/(1-qx+rx^2)=(1-\overline{\mathbf{S}_{q,r}^1}x)/(1-qx+rx^2)$. □

4 | The Horadam $\mathbf{s}_{q,r}$-Split Quaternions

Definition 2. For any integer $n\geq1$, n–th Horadam $\mathbf{s}_{q,r}$–split quaternion is defined as

$$\mathbf{s}_{q,r}^n=w_n+w_{n-1}\mathbf{i}+w_n\mathbf{j}+w_{n-1}\mathbf{k}, \tag{36}$$

where w_n are Horadam numbers with initial conditions $w_0=a$, $w_1=b$, and $\mathbf{i}^2=-1,\mathbf{j}^2=\mathbf{k}^2=\mathbf{ijk}=1$.

It can be easily seen that the sequence $\left\{\mathbf{s}_{q,r}^n\right\}_{n=0}^{\infty}$ satisfies the Horadam recurrence:

$$\mathbf{s}_{q,r}^n=q\mathbf{s}_{q,r}^{n-1}-r\mathbf{s}_{q,r}^{n-2}, \tag{37}$$

with $\mathbf{s}_{q,r}^0=a+((aq-b)/r)\mathbf{i}+a\mathbf{j}+((aq-b)/r)\mathbf{k}$, and $\mathbf{s}_{q,r}^1=b+a\mathbf{i}+b\mathbf{j}+a\mathbf{k}$ is the initial condition.

Theorem 4. *n–th Horadam $\mathbf{s}_{q,r}$–split quaternion is calculated as follows:*

$$\mathbf{s}_{q,r}^n=\mathbf{S}_{q,r}^1\mathbf{s}_{q,r}^{n-1}=w_n+w_{n-1}\mathbf{i}+w_n\mathbf{j}+w_{n-1}\mathbf{k},\quad n>1, \tag{38}$$

with $\mathbf{s}_{q,r}^0=a+((aq-b)/r)\mathbf{i}+a\mathbf{j}+((aq-b)/r)\mathbf{k}$, *and* $\mathbf{s}_{q,r}^1=b+a\mathbf{i}+b\mathbf{j}+a\mathbf{k}$ *is the initial condition.*

Proof 7. We prove it by mathematical induction. For $n=2$, we have

$$\mathbf{s}_{q,r}^2 = \mathbf{S}_{q,r}^1\mathbf{s}_{q,r}^1 = \left(\frac{q}{2} + \frac{1+r}{2}\mathbf{i} + \frac{q}{2}\mathbf{j} + \frac{1-r}{2}\mathbf{k}\right)(b + a\mathbf{i} + b\mathbf{j} + a\mathbf{k})$$
$$= (bq - ar) + b\mathbf{i} + (bq - ar)\mathbf{j} + b\mathbf{k},$$
$$= w_2 + w_1\mathbf{i} + w_2\mathbf{j} + w_1\mathbf{k}. \tag{39}$$

So Equation (38) is true for $n = 2$. Let this be true for $n = k - 1$. Accordingly,

$$\begin{aligned}\mathbf{s}_{q,r}^{k-1} &= \mathbf{S}_{q,r}^1\mathbf{s}_{q,r}^{k-2} \\ &= w_{k-1} + w_{k-2}\mathbf{i} + w_{k-1}\mathbf{j} + w_{k-2}\mathbf{k}.\end{aligned} \tag{40}$$

is true. Now, we show by induction that it is true for $n = k$. Then, we obtain

$$\begin{aligned}\mathbf{s}_{q,r}^k &= \mathbf{S}_{q,r}^1\mathbf{s}_{q,r}^{k-1} \\ &= \left(\frac{q}{2} + \frac{1+r}{2}\mathbf{i} + \frac{q}{2}\mathbf{j} + \frac{1-r}{2}\mathbf{k}\right)(w_{k-1} + w_{k-2}\mathbf{i} + w_{k-1}\mathbf{j} + w_{k-2}\mathbf{k}) \\ &= (qw_{k-1} - rw_{k-2}) + w_{k-1}\mathbf{i} + (qw_{k-1} - rw_{k-2})\mathbf{j} + w_{k-1}\mathbf{k} \\ &= w_k + w_{k-1}\mathbf{i} + w_k\mathbf{j} + w_{k-1}\mathbf{k}.\end{aligned} \tag{41}$$

Thus, by mathematical induction, Equality (38) is true for all positive integers n. □

Theorem 5. *n–th Horadam $\mathbf{s}_{q,r}$-split quaternion is calculated as follows:*

$$\mathbf{s}_{q,r}^n = \mathbf{S}_{q,r}^{n-1}\mathbf{s}_{q,r}^1 = w_n + w_{n-1}\mathbf{i} + w_n\mathbf{j} + w_{n-1}\mathbf{k}, \quad n > 1, \tag{42}$$

with $\mathbf{s}_{q,r}^0 = a + ((aq - b)/r)\mathbf{i} + a\mathbf{j} + ((aq - b)/r)\mathbf{k}$, *and* $\mathbf{s}_{q,r}^1 = b + a\mathbf{i} + b\mathbf{j} + a\mathbf{k}$ *is the initial condition.*

Proof 8. We prove it by mathematical induction. It is clear that Equality (42) is true for $n = 2$. We assume that it is true for $n = k - 1$. That is

$$\mathbf{s}_{q,r}^{k-1} = \mathbf{S}_{q,r}^{k-2}\mathbf{s}_{q,r}^1 = w_{k-1} + w_{k-2}\mathbf{i} + w_{k-1}\mathbf{j} + w_{k-2}\mathbf{k}. \tag{43}$$

We show that it is true for $n = k$. Then, we obtain that

$$\begin{aligned}\mathbf{s}_{q,r}^k &= \mathbf{S}_{q,r}^{k-1}\mathbf{s}_{q,r}^1 = \mathbf{S}_{q,r}^1\left(\mathbf{S}_{q,r}^{k-2}\mathbf{s}_{q,r}^1\right) \\ &= \mathbf{S}_{q,r}^1(w_{k-1} + w_{k-2}\mathbf{i} + w_{k-1}\mathbf{j} + w_{k-2}\mathbf{k}) \\ &= (qw_{k-1} - rw_{k-2}) + w_{k-1}\mathbf{i} + (qw_{k-1} - rw_{k-2})\mathbf{j} + w_{k-1}\mathbf{k} \\ &= w_k + w_{k-1}\mathbf{i} + w_k\mathbf{j} + w_{k-1}\mathbf{k}.\end{aligned} \tag{44}$$

By mathematical induction, Equality (42) is true for all positive integers n. □

Theorem 6 (Binet-like identity). *Binet-like formula for Horadam $\mathbf{s}_{q,r}$-split quaternions is*

$$\mathbf{s}_{q,r}^n = \alpha_1\underline{\alpha_1}\,x_1^n + \alpha_2\underline{\alpha_2}\,x_2^n, \tag{45}$$

where $\alpha_1 = (b - ax_2)/(x_1 - x_2)$, $\alpha_2 = (ax_1 - b)/(x_1 - x_2)$, *and* x_1, x_2 *are roots of the equation* $x^2 - qx + r = 0$ *and* $\underline{\alpha_1} = 1 + x_1^{-1}\mathbf{i} + \mathbf{j} + x_1^{-1}\mathbf{k}$, $\underline{\alpha_2} = 1 + x_2^{-1}\mathbf{i} + \mathbf{j} + x_2^{-1}\mathbf{k}$.

Proof 9. We write $\alpha_1x_1^{n-1} + \alpha_2x_2^{n-1}$, $\alpha_1x_1^n + \alpha_2x_2^n$ instead of w_{n-1}, w_n in Equation (36), respectively. When the necessary arrangements are made, we obtain

$$\begin{aligned}\mathbf{s}_{q,r}^n &= w_n + w_{n-1}\mathbf{i} + w_n\mathbf{j} + w_{n-1}\mathbf{k} \\ &= \left(\alpha_1x_1^n + \alpha_2x_2^n\right) + \left(\alpha_1x_1^{n-1} + \alpha_2x_2^{n-1}\right)\mathbf{i} + \\ &\quad + \left(\alpha_1x_1^n + \alpha_2x_2^n\right)\mathbf{j} + \left(\alpha_1x_1^{n-1} + \alpha_2x_2^{n-1}\right)\mathbf{k} \\ &= \left(\alpha_1x_1^n + \alpha_1x_1^{n-1}\mathbf{i} + \alpha_1x_1^n\mathbf{j} + \alpha_1x_1^{n-1}\mathbf{k}\right) + \\ &\quad + \left(\alpha_2x_2^n + \alpha_2x_2^{n-1}\mathbf{i} + \alpha_2x_2^n\mathbf{j} + \alpha_2x_2^{n-1}\mathbf{k}\right).\end{aligned} \tag{46}$$

Thus,

$$\mathbf{s}_{q,r}^n = \alpha_1x_1^n\left(1 + x_1^{-1}\mathbf{i} + \mathbf{j} + x_1^{-1}\mathbf{k}\right) + \alpha_2x_2^{n-1}\left(1 + x_2^{-1}\mathbf{i} + \mathbf{j} + x_2^{-1}\mathbf{k}\right), \tag{47}$$

can be written. Thus, we have $\mathbf{s}_{q,r}^n = \alpha_1\underline{\alpha_1}\,x_1^n + \alpha_2\underline{\alpha_2}\,x_2^n$. □

Theorem 7. *Generating function of Horadam $\mathbf{s}_{q,r}$-split quaternions is*

$$c_{q,r}(x) = \frac{\mathbf{s}_{q,r}^0 + \left(\mathbf{s}_{q,r}^1 - q\mathbf{s}_{q,r}^0\right)x}{1 - qx + rx^2}, \tag{48}$$

where $\mathbf{s}_{q,r}^0 = a + ((aq - b)/r)\mathbf{i} + a\mathbf{j} + ((aq - b)/r)\mathbf{k}$, *and* $\mathbf{s}_{q,r}^1 = b + a\mathbf{i} + b\mathbf{j} + a\mathbf{k}$ *is the initial condition.*

Proof 10. Let the function

$$c_{q,r}(x) = \sum_{n=0}^{\infty}\mathbf{s}_{q,r}^n x^n, \tag{49}$$

be the generating function of $\mathbf{s}_{q,r}^n$. Multiplying both sides of (49) by $-qx$ and rx^2, respectively, yields the following three equations:

$$\begin{aligned}c_{q,r}(x) &= \mathbf{s}_{q,r}^0 + \mathbf{s}_{q,r}^1x + \mathbf{s}_{q,r}^2x^2 + \mathbf{s}_{q,r}^3x^3 + \cdots \\ -qxc_{q,r}(x) &= qx\mathbf{s}_{q,r}^0 - q\mathbf{s}_{q,r}^1x^2 - q\mathbf{s}_{q,r}^2x^3 - q\mathbf{s}_{q,r}^3x^4 - \cdots \\ rx^2c_{q,r}(x) &= r\mathbf{s}_{q,r}^0x^2 + r\mathbf{s}_{q,r}^1x^3 + r\mathbf{s}_{q,r}^2x^4 + r\mathbf{s}_{q,r}^3x^5 + \cdots.\end{aligned} \tag{50}$$

These equations can be arranged as follows, so

$$c_{q,r}(x) = \frac{\sum_{n=0}^{\infty}\mathbf{s}_{q,r}^n x^n - \left(q\sum_{n=0}^{\infty}\mathbf{s}_{q,r}^n x^{n+1} - r\sum_{n=0}^{\infty}\mathbf{s}_{q,r}^n x^{n+2}\right)}{1 - qx + rx^2},$$

$$c_{q,r}(x) = \frac{\mathbf{s}_{q,r}^0 + \mathbf{s}_{q,r}^1x - q\mathbf{s}_{q,r}^0x + \sum_{n=2}^{\infty}\left(\mathbf{s}_{q,r}^n - \left(q\mathbf{s}_{q,r}^{n-1} - r\mathbf{s}_{q,r}^{n-2}\right)\right)x^n}{1 - qx + rx^2}, \tag{51}$$

is calculated. Using (37), we have $\sum_{n=2}^{\infty}(\mathbf{s}_{q,r}^n - (q\mathbf{s}_{q,r}^{n-1} - r\mathbf{s}_{q,r}^{n-2}))x^n = 0$. From here, it is obtained that $c_{q,r}(x) = (\mathbf{s}_{q,r}^0 + (\mathbf{s}_{q,r}^1 - q\mathbf{s}_{q,r}^0)x)/(1 - qx + rx^2)$. □

Theorem 8 (Cassini-like identity). *For* $n \in \mathbb{N}$,

$$\mathbf{s}_{q,r}^{n+1}\mathbf{s}_{q,r}^{n-1} - \left(\mathbf{s}_{q,r}^n\right)^2 = -2r^{n-1}\rho^2\mathbf{v}_1, \tag{52}$$

$$\mathbf{s}_{q,r}^{n-1}\mathbf{s}_{q,r}^{n+1} - \left(\mathbf{s}_{q,r}^n\right)^2 = -2r^{n-2}\rho^2\mathbf{v}_2, \tag{53}$$

where $\mathbf{v}_1 = 1 + \mathbf{j}$, $\mathbf{v}_2 = r + q\mathbf{i} + r\mathbf{j} + q\mathbf{k}$.

Proof 11. Using Definition 2 and multiplication of split quaternions, when the calculation is made for the left hand side (L) of Equation (52), it can be seen that

$$\begin{aligned} L &= \left(w_{n+1} + w_n\mathbf{i} + w_{n+1}\mathbf{j} + w_n\mathbf{k}\right)\left(w_{n-1} + w_{n-2}\mathbf{i} + w_{n-1}\mathbf{j} + w_{n-2}\mathbf{k}\right) \\ &\quad - \left(w_n + w_{n-1}\mathbf{i} + w_n\mathbf{j} + w_{n-1}\mathbf{k}\right)^2 \\ &= 2w_{n-1}\left(w_{n+1} + w_n\mathbf{i} + w_{n+1}\mathbf{j} + w_n\mathbf{k}\right) \\ &\quad - 2w_n\left(w_n + w_{n-1}\mathbf{i} + w_n\mathbf{j} + w_{n-1}\mathbf{k}\right) \\ &= 2\left(\left(w_{n-1}w_{n+1} - w_n^2\right) + \left(w_{n-1}w_{n+1} - w_n^2\right)\mathbf{j}\right) \\ &= 2\left(w_{n-1}w_{n+1} - w_n^2\right)(1+\mathbf{j}). \end{aligned} \tag{54}$$

From Equation (15), we obtain $\mathbf{s}_{q,r}^{n+1}\mathbf{s}_{q,r}^{n-1} - \mathbf{s}_{q,r}^{n} = -2r^{n-1}\rho^2\mathbf{v}_1$. Equation (53) can be proven in a similar way. □

Definition 3. For any integer $n > 1$, $n-$th Horadam $\mathbf{P}_{q,r}-$ pure split quaternion is defined as

$$\mathbf{P}_{q,r}^n = \frac{w_{n-1} + w_{n+1}}{2}\mathbf{i} + w_n\mathbf{j} + \frac{w_{n-1} - w_{n+1}}{2}\mathbf{k}, \quad n > 1, \tag{55}$$

where w_n denotes Horadam numbers with initial conditions $w_0 = a$, $w_1 = b$, and $\mathbf{i}^2 = -1, \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = 1$.

This expression is obtained from the structure of the Horadam $\mathbf{s}_{q,r}$-split quaternion by eliminating the scalar component and arranging the remaining vector part in terms of the neighboring Horadam numbers w_{n-1}, w_n, and w_{n+1}.

It can be easily seen that the sequence $\left\{\mathbf{P}_{q,r}^n\right\}_{n=0}^{\infty}$ satisfies the Horadam recurrence:

$$\mathbf{P}_{q,r}^n = q\mathbf{P}_{q,r}^{n-1} - r\mathbf{P}_{q,r}^{n-2}, \tag{56}$$

with $\mathbf{P}_{q,r}^0 = ((b + (aq - b)/r)/2)\mathbf{i} + a\mathbf{j} + (((aq - b)/r - b)/2)\mathbf{k}$, and $\mathbf{P}_{q,r}^1 = ((a + qb - ra)/2)\mathbf{i} + b\mathbf{j} + (a - (qb - ra)/2)\mathbf{k}$ is the initial condition.

Theorem 9. *$n-$th Horadam $\mathbf{P}_{q,r}-$pure split quaternion is calculated as follows: for $n > 1$,*

$$\mathbf{P}_{q,r}^n = \mathbf{S}_{q,r}^1\mathbf{P}_{q,r}^{n-1} = \frac{w_{n-1} + w_{n+1}}{2}\mathbf{i} + w_n\mathbf{j} + \frac{w_{n-1} - w_{n+1}}{2}\mathbf{k}, \tag{57}$$

$$\mathbf{P}_{q,r}^n = \mathbf{S}_{q,r}^{n-1}\mathbf{P}_{q,r}^1 = \frac{w_{n-1} + w_{n+1}}{2}\mathbf{i} + w_n\mathbf{j} + \frac{w_{n-1} - w_{n+1}}{2}\mathbf{k}, \tag{58}$$

with $\mathbf{P}_{q,r}^0 = ((b + (aq - b)/r)/2)\mathbf{i} + a\mathbf{j} + (((aq - b)/r - b)/2)\mathbf{k}$, and $\mathbf{P}_{q,r}^1 = ((a + qb - ra)/2)\mathbf{i} + b\mathbf{j} + (a - (qb - ra)/2)\mathbf{k}$ is the initial condition.

Proof 12. It can be proved as in Theorems 4 and 5. □

Theorem 10 (Binet-like identity). *Binet-like formula for Horadam $\mathbf{P}_{q,r}$-split quaternions is*

$$\mathbf{P}_{q,r}^n = \alpha_1\widetilde{\alpha_1}x_1^n + \alpha_2\widetilde{\alpha_2}x_2^n, \tag{59}$$

where $\alpha_1 = (b - ax_2)/(x_1 - x_2)$, $\alpha_2 = (ax_1 - b)/(x_1 - x_2)$, and x_1, x_2 are roots of the equation $x^2 - qx + r = 0$ and $\widetilde{\alpha_1} = (1 + x_1^2)/(2x_1)\mathbf{i} + \mathbf{j} + ((1 - x_1^2)/(2x_1))\mathbf{k}$, $\widetilde{\alpha_2} = (1 + x_2^2)/(2x_2)\mathbf{i} + \mathbf{j} + ((1 - x_2^2)/(2x_2))\mathbf{k}$.

Proof 13. We write $\alpha_1x_1^{n-1} + \alpha_2x_2^{n-1}$, $\alpha_1x_1^n + \alpha_2x_2^n$, and $\alpha_1x_1^{n+1} + \alpha_2x_2^{n+1}$ instead of w_{n-1}, w_n, and w_{n+1} in Equation (55), respectively. After computation, we find that

$$\begin{aligned} \mathbf{P}_{q,r}^n &= \frac{w_{n-1} + w_{n+1}}{2}\mathbf{i} + w_n\mathbf{j} + \frac{w_{n-1} - w_{n+1}}{2}\mathbf{k} \\ &= \frac{\left(\alpha_1x_1^{n-1} + \alpha_2x_2^{n-1}\right) + \left(\alpha_1x_1^{n+1} + \alpha_2x_2^{n+1}\right)}{2}\mathbf{i} + \\ &\quad + \left(\alpha_1x_1^n + \alpha_2x_2^n\right)\mathbf{j} + \frac{\left(\alpha_1x_1^{n-1} + \alpha_2x_2^{n-1}\right) - \left(\alpha_1x_1^{n+1} + \alpha_2x_2^{n+1}\right)}{2}\mathbf{k} \\ &= \alpha_1x_1^n\left(\frac{x_1^{-1} + x_1}{2}\mathbf{i} + \mathbf{j} + \frac{x_1^{-1} - x_1}{2}\mathbf{k}\right) + \\ &\quad + \alpha_2x_2^n\left(\frac{x_2^{-1} + x_2}{2}\mathbf{i} + \mathbf{j} + \frac{x_2^{-1} - x_2}{2}\mathbf{k}\right). \end{aligned} \tag{60}$$

Thus,

$$\begin{aligned} \mathbf{P}_{q,r}^n &= \alpha_1x_1^n\left(\frac{1 + x_1^2}{2x_1}\mathbf{i} + \mathbf{j} + \frac{1 - x_1^2}{2x_1}\mathbf{k}\right) \\ &\quad + \alpha_2x_2^{n-1}\left(\frac{1 + x_2^2}{2x_2}\mathbf{i} + \mathbf{j} + \frac{1 - x_2^2}{2x_2}\mathbf{k}\right), \end{aligned} \tag{61}$$

is obtained. It can be seen that $\mathbf{P}_{q,r}^n = \alpha_1\widetilde{\alpha_1}x_1^n + \alpha_2\widetilde{\alpha_2}x_2^n$. □

Theorem 11. *Generating function of Horadam $\mathbf{P}_{q,r}$-split quaternions is*

$$P_{q,r}(x) = \frac{\mathbf{P}_{q,r}^0 + \left(\mathbf{P}_{q,r}^1 - q\mathbf{P}_{q,r}^0\right)x}{\left(1 - qx + rx^2\right)}, \tag{62}$$

where $\mathbf{P}_{q,r}^0 = ((b + (aq - b)/r)/2)\mathbf{i} + a\mathbf{j} + (((aq - b)/r - b)/2)\mathbf{k}$, and $\mathbf{P}_{q,r}^1 = ((a + qb - ra)/2)\mathbf{i} + b\mathbf{j} + (a - (qb - ra)/2)\mathbf{k}$ is the initial condition.

Proof 14. It can be proved as in Theorem 7. □

Theorem 12 (Cassini-like identity). *For $n \in \mathbb{N}$,*

$$\mathbf{P}_{q,r}^{n+1}\mathbf{P}_{q,r}^{n-1} - \left(\mathbf{P}_{q,r}^n\right)^2 = r^{n-1}\rho^2\mathbf{u}, \tag{63}$$

$$\mathbf{P}_{q,r}^{n-1}\mathbf{P}_{q,r}^{n+1} - \left(\mathbf{P}_{q,r}^n\right)^2 = r^{n-1}\rho^2\overline{\mathbf{u}}, \tag{64}$$

where $\mathbf{u} = -(1/(2r))(4r - q^2) + (1/2)(q/r)(r + 1)\mathbf{i} + (1/2)(q^2/r)\mathbf{j} - (1/2)(q/r)(r - 1)\mathbf{k}$.

Proof 15. Using Equation (55) and the equation

$$\begin{aligned} w_{n-2} &= \frac{w_n - qw_{n-1}}{-r}, \\ w_{n+1} &= qw_n - rw_{n-1}, \\ w_{n+2} &= q^2w_n - qrw_{n-1} - rw_n, \end{aligned} \tag{65}$$

when the calculation is made for the left hand side (L) of Equation (63), it can be seen that

$$
\begin{aligned}
L &= \left(\frac{w_n + w_{n+2}}{2}\mathbf{i} + w_{n+1}\mathbf{j} + \frac{w_n - w_{n+2}}{2}\mathbf{k}\right)\left(\frac{w_{n-2} + w_n}{2}\mathbf{i} + w_{n-1}\mathbf{j} + \frac{w_{n-2} - w_n}{2}\mathbf{k}\right) \\
&\quad - \left(\frac{w_{n-1} + w_{n+1}}{2}\mathbf{i} + w_n\mathbf{j} + \frac{w_{n-1} - w_{n+1}}{2}\mathbf{k}\right)^2 \\
&= \frac{q^2 - 4r}{2r}\left(w_n^2 - qw_nw_{n-1} + rw_{n-1}^2\right) \\
&\quad + \frac{q(r+1)}{2r}\left(w_n^2 - qw_nw_{n-1} + rw_{n-1}^2\right)\mathbf{i} \\
&\quad + \frac{q^2}{2r}\left(w_n^2 - qw_nw_{n-1} + rw_{n-1}^2\right)\mathbf{j} \\
&\quad - \frac{q(r-1)}{2r}\left(w_n^2 - qw_nw_{n-1} + rw_{n-1}^2\right)\mathbf{k}.
\end{aligned} \tag{66}
$$

Since $w_{n+1} = qw_n - rw_{n-1}$ and from Equation (15), we obtain $\mathbf{P}_{q,r}^{n+1}\mathbf{P}_{q,r}^{n-1} - \mathbf{P}_{q,r}^{n} = r^{n-1}\rho^2\mathbf{u}$. Equation (64) can be proven in a similar way. □

5 | Conclusions

In this study, the structural relationship between Horadam numbers and the split quaternion algebra was established, based on the observation that both possess a 2×2 real matrix representation. Within this framework, the Fibonacci $\mathbf{S}_{q,r}$-split quaternions and the Horadam $\mathbf{s}_{q,r}$-split quaternions were defined. Furthermore, several fundamental results, including the Binet-like formula, generating function, and Cassini-like identities, were derived to reveal the algebraic properties of these sequences. The investigation of Horadam sequences over the split quaternion algebra enhances their potential applications in number theory and hypercomplex algebra. Consequently, this work provides a theoretical framework that may inspire further research in algebraic geometry and applied mathematics, offering a new perspective on the interaction between Horadam numbers and split quaternion structures. A possible direction for future research is the investigation of the geometric interpretation of these structures in the context of the hyperbolic geometry associated with the split quaternion algebra.

Compared with other classical polynomial families such as Chebyshev, Legendre, or Jacobi polynomials, Horadam-type structures provide a flexible four-parameter recurrence that simultaneously generalizes several well-known sequences, including the Fibonacci and Lucas numbers. This flexibility makes them particularly suitable for algebraic constructions involving matrix representations and hypercomplex extensions, such as the split quaternion framework considered in this work.

Author Contributions

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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