

Research Article

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A new frame for non-lightlike curves in Minkowski 3-space: hybrid frame

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Abstract: In this study, we introduce a hybrid frame consisting of elliptic, hyperbolic, and parabolic vectors for the study of non-lightlike curves in Minkowski 3-space. In contrast to classical approaches that treat different causal characters separately, the proposed frame allows these vector types to be considered within a single framework. We define the curvature and torsion functions with respect to the hybrid frame and examine their relations to the corresponding Frenet invariants. In addition, we investigate the osculating spheres of non-lightlike curves using this frame and obtain characterizations of their centers. The results provide an alternative formulation for the geometric analysis of non-lightlike curves and contribute to the existing theory of curves in Minkowski space.

Keywords: Frenet frame; Bishop frame; quasi frame; Cartan frame; hybrid numbers

MSC 2020: 14H50; 15A63; 53A04; 53A35; 53B30; 70B10; 70E17

1 Introduction

In differential geometry, the study of curves in Euclidean and semi-Riemannian spaces has long been a fundamental topic, particularly in understanding their behavior under various transformations and motions. In Euclidean space, the Frenet frame, which provides an orthonormal basis along a curve, has been widely used as a fundamental tool for analyzing intrinsic geometric properties such as curvature and torsion. However, certain limitations of the Frenet frame arise when dealing with curves in spaces of different geometric structures (e.g., semi-Riemannian spaces) or in cases where the curvature vanishes. As a result, alternative frames such as the Bishop frame and the Cartan frame have been employed [1–4].

A common feature of the Frenet, Bishop, and Cartan frames is that they are constructed according to specific geometric conditions adapted to particular causal types of curves. In Minkowski space, however, the coexistence of elliptic (timelike), hyperbolic (spacelike), and parabolic (lightlike) behaviors suggests the need for a framework capable of reflecting these different structures in a unified manner. This observation motivates the search for an alternative frame formulation that can simultaneously incorporate these vector characteristics without separating them into distinct cases.

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The Bishop frame extends the Frenet frame and is particularly useful when the curvature function is identically zero [1]. Comparative studies of the Bishop and Frenet frames have been carried out extensively in Euclidean 3-space [1–4]. On the other hand, the Cartan frame was introduced by Bonnor [5] to examine the behavior of null curves in Minkowski spacetime and to establish fundamental existence and congruence theorems, which were further explored in [6].

Alternative framed approaches have also been considered in recent studies. For example, Pei et al. [7] investigated pseudo-circular evolutes and involutes of lightcone framed curves in Lorentz–Minkowski 3-space, while Atalay and Murat [8] studied families of parametric and ruled surfaces according to the Bishop frame.

In this study, we introduce an alternative frame based on hybrid numbers, which form a Clifford algebra isomorphic to split quaternions. Hybrid numbers provide a novel perspective by incorporating elliptic, hyperbolic, and parabolic vector characteristics. Motivated by this observation, we define a hybrid frame spanned by the complex vector $\mathbf{E}$ with $\mathbf{E}^2 = -1$, the hyperbolic vector $\mathbf{H}$ with $\mathbf{H}^2 = 1$, and the parabolic vector $\mathbf{P}$ with $\mathbf{P}^2 = 0$. The hybrid frame is constructed in Lorentz–Minkowski space and satisfies the relation $\mathbf{EH} = -\mathbf{HE} = \mathbf{E} + \mathbf{P}$. By introducing this frame, our goal is to contribute to the study of non-lightlike curves in Minkowski space and to provide an alternative geometric viewpoint for their analysis.

Recent studies have also demonstrated that hybrid numbers provide an effective tool for modeling geometric transformations in generalized number planes. In particular, Öztürk et al. [9] investigated inversion transformations with respect to conics in hybrid number planes, further highlighting the geometric versatility of hybrid units. Moreover, several recent works have expanded the algebraic and geometric structure of hybrid numbers, including new polar representations introduced by Koçay et al. [10] and the characterization of null hybrid curves and hybrid Bertrand curves examined by Alo [11]. These contributions illustrate the growing mathematical interest in hybrid number structures and their applications.

Within this framework, we investigate the curvature and torsion of curves in the hybrid frame. Based on these fundamental invariants, we analyze the osculating spheres of curves in Minkowski space using the hybrid frame approach. The osculating sphere is a significant concept in differential geometry, providing insight into the local geometric properties of a curve and its relation to curvature. Özdemir examined the osculating sphere of non-lightlike curves using the parallel frame [2]. Additionally, Soytürk et al. studied osculating spheres for both lightlike and non-lightlike curves in semi-Riemannian spaces, specifically considering the spaces $\mathbb{R}_1^3$, $\mathbb{R}_1^4$, and $\mathbb{R}_2^4$ [12]. Furthermore, the concept of the osculating k -sphere in n -dimensional Euclidean space was explored by Fudali [13].

This article is structured as follows. In the first section, we present the fundamental concepts of Lorentz–Minkowski space, hybrid numbers, and the hybrid multiplication tables. In the second section, we define the hybrid frame spanned by the set $\{\mathbf{E}, \mathbf{H}, \mathbf{P}\}$. Finally, in the third section, we examine the osculating spheres of non-lightlike curves using the hybrid frame approach.

The results obtained in this study contribute to the geometric theory of non-lightlike curves in Minkowski space and provide an alternative framework for their analysis.

2 Preliminaries

Hybrid numbers have recently attracted attention from many mathematicians, particularly due to their number-theoretic, matrix, and geometric properties. One of their prominent features is that they unify complex, dual, and hyperbolic numbers [14–18].

Definition 2.1. The set of hybrid numbers, denoted by $\mathbb{K}$, is defined as

$$\mathbb{K} = \left\{ a + b\mathbf{i} + c\epsilon + d\mathbf{h} : a, b, c, d \in \mathbb{R}, \mathbf{i}^2 = -1, \epsilon^2 = 0, \mathbf{h}^2 = 1, \mathbf{i}\mathbf{h} = -\mathbf{h}\mathbf{i} = \epsilon + \mathbf{i} \right\}.$$

Since the product of the hybrid units is defined by $\mathbf{i}^2 = -1$, $\varepsilon^2 = 0$, $\mathbf{h}^2 = 1$, and $\mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i}$, the two-dimensional Clifford algebras $\mathcal{C}\ell(-1)$, $\mathcal{C}\ell(0)$, and $\mathcal{C}\ell(1)$ are spanned by $\text{span}\{1, \mathbf{i}\}$, $\text{span}\{1, \varepsilon\}$, and $\text{span}\{1, \mathbf{h}\}$, respectively. The products of the hybrid units are given in the following table:

$\cdot$	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	-1	$1 - \mathbf{h}$	$\varepsilon + \mathbf{i}$
ε	$\mathbf{h} + 1$	0	$-\varepsilon$
$\mathbf{h}$	$-\varepsilon - \mathbf{i}$	ε	1

In this table, the units $\mathbf{i}$, ε , and $\mathbf{h}$ correspond to the elliptic, parabolic, and hyperbolic units, respectively [14–18].

Minkowski 3-space is the Euclidean 3-space equipped with the Lorentzian inner product $\langle \mathbf{v}, \mathbf{w} \rangle_L = -v_1w_1 + v_2w_2 + v_3w_3$, where $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3) \in \mathbb{E}^3$. In $\mathbb{E}_1^3$, any vector $\mathbf{v}$ is called timelike, lightlike or spacelike, if $\langle \mathbf{v}, \mathbf{v} \rangle_L < 0$, $\langle \mathbf{v}, \mathbf{v} \rangle_L = 0$, $\langle \mathbf{v}, \mathbf{v} \rangle_L > 0$, respectively. However, in this study, we refer to these vectors as elliptic, parabolic, and hyperbolic vectors, respectively, inspired by hybrid numbers. Accordingly, elliptic, hyperbolic, and parabolic vectors correspond to timelike, spacelike, and lightlike vectors, respectively, in the Lorentzian setting. The norm of the vector $\mathbf{v}$ is defined by $\|\mathbf{v}\| = \sqrt{|\langle \mathbf{v}, \mathbf{v} \rangle_L|}$. The Lorentzian vector product

$$\mathbf{v} \times_L \mathbf{w} = (v_3w_2 - v_2w_3, -v_1w_3 + v_3w_1, v_1w_2 - v_2w_1),$$

where $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$.

Throughout this paper, the Lorentzian scalar product is denoted by $\langle \cdot, \cdot \rangle_L$, while the Lorentzian vector product is denoted by $\times_L$. These two operations will be used consistently in order to avoid any ambiguity in the geometric computations.

We now recall the formulas and Frenet frames in the Minkowski space $\mathbb{R}^3$, where the metric is $\langle \cdot, \cdot \rangle_L = -dx^2 + dy^2 + dz^2$. Locally, a curve $\gamma(s)$ in $\mathbb{E}_1^3$ may have one of the following causal characters: If $\langle \gamma', \gamma' \rangle_L < 0$, $\langle \gamma', \gamma' \rangle_L = 0$ or $\langle \gamma', \gamma' \rangle_L > 0$, then the curve γ is called timelike, lightlike or spacelike, respectively. The Frenet formulas for any non-lightlike curves can be summarized as follows:

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\delta_{21}\kappa & 0 & \tau \\ 0 & -\delta_{32}\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}$$

where $\delta_i = \langle v_i, v_i \rangle_L$ and $\delta_{ij} = \delta_i \delta_j$.

3 {E, H, P} frame

We can span Minkowski space with an elliptic (**E**), a hyperbolic (**H**), and a parabolic (**P**) vector. The frame constructed with **E**, **H**, **P** vectors is called a hybrid frame. The scalar product and vector product tables of **E**, **H**, **P** are given by

$$\left| \begin{array}{c|ccc} \langle \cdot, \cdot \rangle & \mathbf{E} & \mathbf{H} & \mathbf{P} \\ \hline \mathbf{E} & -1 & 0 & 1 \\ \mathbf{H} & 0 & 1 & 0 \\ \mathbf{P} & 1 & 0 & 0 \end{array} \right| \quad \text{and} \quad \left| \begin{array}{c|ccc} \times & \mathbf{E} & \mathbf{H} & \mathbf{P} \\ \hline \mathbf{E} & 0 & \mathbf{P} + \mathbf{E} & -\mathbf{H} \\ \mathbf{H} & -\mathbf{E} - \mathbf{P} & 0 & \mathbf{P} \\ \mathbf{P} & \mathbf{H} & -\mathbf{P} & 0 \end{array} \right|,$$

respectively. These three vectors form a non-orthogonal basis in the 3D Lorentz space. Thus, a new frame is obtained that includes all types of vectors to examine timelike, spacelike and lightlike curves. This frame differs from other frames in that it includes all three types of vectors in Lorentz space. Since

$$\det(\mathbf{E}, \mathbf{H}, \mathbf{P}) = \langle \mathbf{E} \times \mathbf{H}, \mathbf{P} \rangle = \langle \mathbf{P} + \mathbf{E}, \mathbf{P} \rangle = 1$$

$$\det(\mathbf{E}, \mathbf{H}, \mathbf{P}) = \langle \mathbf{E}, \mathbf{H} \times \mathbf{P} \rangle = \langle \mathbf{E}, \mathbf{P} \rangle = 1$$

this frame is positively oriented in Lorentz space.

Definition 3.1. Curvature measures how a curve changes direction at a given point. Let $\gamma : I \rightarrow \mathbb{R}_1^3$ be a unit speed curve with the frame $\{\mathbf{E}, \mathbf{H}, \mathbf{P}\}$ in the Lorentzian space. Then, the functions

$$k_1, k_2, k_3 : I \rightarrow \mathbb{R}$$

defined as

$$\kappa_1 = \langle \mathbf{E}', \mathbf{H} \rangle, \quad \kappa_2 = \langle \mathbf{E}', \mathbf{P} \rangle, \quad \text{and} \quad \kappa_3 = \langle \mathbf{H}', \mathbf{P} \rangle$$

are called curvature functions of the curve γ .

Remark 3.2. Since $\{\mathbf{E}, \mathbf{H}, \mathbf{P}\}$ forms a basis of Minkowski space, the derivatives of the frame vectors can be expressed as linear combinations of the frame itself. Writing

$$\mathbf{E}' = a_1 \mathbf{E} + a_2 \mathbf{H} + a_3 \mathbf{P}, \quad \mathbf{H}' = b_1 \mathbf{E} + b_2 \mathbf{H} + b_3 \mathbf{P},$$

the coefficients are obtained by taking the scalar product with the corresponding frame vectors. In particular,

$$\kappa_1 = \langle \mathbf{E}', \mathbf{H} \rangle, \quad \kappa_2 = \langle \mathbf{E}', \mathbf{P} \rangle, \quad \kappa_3 = \langle \mathbf{H}', \mathbf{P} \rangle.$$

These functions are not independent new invariants of the curve. They are directly related to the classical curvature and torsion, written with respect to the hybrid frame.

3.1 Timelike curves

Definition 3.3. Let $\gamma : I \rightarrow \mathbb{E}_1^3$ be a timelike curve depending on the arclength parameter s . This means that the frame $\{\mathbf{E}, \mathbf{H}, \mathbf{P}\}$ is uniquely determined where

$$\mathbf{E} = \gamma'(s), \quad \mathbf{H} = \frac{\gamma''(s)}{\|\gamma''(s)\|} \quad \text{and} \quad \mathbf{P} = \mathbf{E} \times \mathbf{H} - \mathbf{E}.$$

This frame is called the hybrid frame of the timelike curve. Since

$$\det(\mathbf{E}, \mathbf{H}, \mathbf{P}) = \langle \mathbf{E} \times \mathbf{H}, \mathbf{P} \rangle = \langle \mathbf{P} + \mathbf{E}, \mathbf{P} \rangle = 1$$

the frame $\{\mathbf{E}, \mathbf{H}, \mathbf{P}\}$ for the timelike curve is a positively directed basis.

Theorem 3.4. Let $\gamma : I \rightarrow \mathbb{E}_1^3$ be a timelike curve depending on the arclength parameter s . The hybrid frame formulas are

$$\begin{bmatrix} \mathbf{E}' \\ \mathbf{H}' \\ \mathbf{P}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 \\ \kappa_3 & 0 & \kappa_3 - \kappa_1 \\ 0 & -\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa + \tau & 0 & \tau \\ 0 & -\kappa - \tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \\ \mathbf{P} \end{bmatrix}. \quad (1)$$

Proof. The relationship between the hybrid frame vectors $\mathbf{E}, \mathbf{H}, \mathbf{P}$ and the Frenet vectors $\mathbf{T}, \mathbf{N}, \mathbf{B}$ of a timelike curve can be given as follows:

$$\begin{bmatrix} \mathbf{E} \\ \mathbf{H} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}$$

Thus, we can write

$$\mathbf{E} = \mathbf{T}, \mathbf{H} = \mathbf{N}$$

and

$$\mathbf{E} \times \mathbf{H} = \mathbf{P} + \mathbf{E} \Rightarrow \mathbf{B} = \mathbf{P} + \mathbf{T} \Rightarrow \mathbf{P} = \mathbf{B} - \mathbf{T}.$$

Moreover, the relationships between the curvatures of the curve γ are given by

$$\kappa_1 = \langle \mathbf{E}', \mathbf{H} \rangle = \langle \mathbf{T}', \mathbf{N} \rangle = \langle \kappa \mathbf{N}, \mathbf{N} \rangle = \kappa,$$

$$\kappa_2 = \langle \mathbf{E}', \mathbf{P} \rangle = \langle \mathbf{T}', \mathbf{B} - \mathbf{T} \rangle = \langle \kappa \mathbf{N}, \mathbf{B} - \mathbf{T} \rangle = 0,$$

$$\kappa_3 = \langle \mathbf{H}', \mathbf{P} \rangle = \langle \mathbf{N}', \mathbf{B} - \mathbf{T} \rangle = \langle \kappa \mathbf{T} + \tau \mathbf{B}, \mathbf{B} - \mathbf{T} \rangle = \kappa + \tau.$$

Therefore, for a timelike curve the hybrid frame formulas can be calculated as follows.

If $\mathbf{E}' = a_1 \mathbf{E} + b_1 \mathbf{H} + c_1 \mathbf{P}$, then

$$\langle \mathbf{E}', \mathbf{E} \rangle = \langle a_1 \mathbf{E} + b_1 \mathbf{H} + c_1 \mathbf{P}, \mathbf{E} \rangle = -a_1 + c_1 = 0,$$

$$\langle \mathbf{E}', \mathbf{P} \rangle = \langle a_1 \mathbf{E} + b_1 \mathbf{H} + c_1 \mathbf{P}, \mathbf{P} \rangle = a_1 = \kappa_2 = 0,$$

$$\langle \mathbf{E}', \mathbf{H} \rangle = \langle a_1 \mathbf{E} + b_1 \mathbf{H} + c_1 \mathbf{P}, \mathbf{H} \rangle = b_1 = \kappa_1.$$

Similarly, if $\mathbf{H}' = a_2 \mathbf{E} + b_2 \mathbf{H} + c_2 \mathbf{P}$ and $\mathbf{P}' = a_3 \mathbf{E} + b_3 \mathbf{H} + c_3 \mathbf{P}$, we can obtain

$$-a_2 + c_2 = -\kappa_1,$$

$$a_2 = \kappa_3,$$

$$b_2 = 0,$$

and

$$-a_3 + c_3 = -\kappa_2 = 0,$$

$$a_3 = 0,$$

$$b_3 = -\kappa_3.$$

So, we get the matrix equality (1). □

Remark 3.5. Equation (1) shows that the hybrid frame evolves according to the curvature functions κ_1 , κ_2 , and κ_3 , which are directly related to the classical curvature and torsion of the curve. In this way, the hybrid formulation provides an alternative formulation of the Frenet motion in Minkowski space.

The geometric relationship between the hybrid frame vectors and the Frenet frame for timelike curves is illustrated in Figure 1.

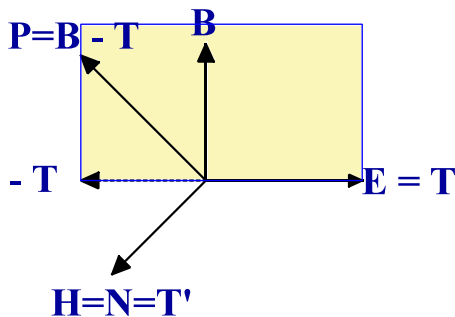


Figure 1: The relationship between the hybrid frame vectors $\mathbf{E}$, $\mathbf{H}$, $\mathbf{P}$ and the Frenet vectors $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ of a timelike curve.

Example. Let's construct the frame $\{\mathbf{E}, \mathbf{H}, \mathbf{P}\}$ of the unit speed timelike curve $\gamma(s) = (\sqrt{2}s, \cos s, \sin s)$ and compute its curvatures and hybrid frame formulas. The vectors of the frame $\{\mathbf{E}, \mathbf{H}, \mathbf{P}\}$ of the curve $\gamma(s)$ are

$$\begin{aligned}\mathbf{E} &= (\sqrt{2}, -\sin s, \cos s), \\ \mathbf{H} &= (0, -\cos s, -\sin s), \\ \mathbf{P} &= \mathbf{E} \times \mathbf{H} - \mathbf{E} = (\sqrt{2} + 1)(-1, \sin s, -\cos s).\end{aligned}$$

The curvatures of the curve $\gamma(s)$ are found as follows:

$$\begin{aligned}\kappa_1 &= \langle \mathbf{E}', \mathbf{H} \rangle = 1 \\ \kappa_2 &= \langle \mathbf{E}', \mathbf{P} \rangle = 0 \\ \kappa_3 &= \langle \mathbf{H}', \mathbf{P} \rangle = \sqrt{2} + 1\end{aligned}$$

The hybrid frame formulas are found as follows:

$$\begin{bmatrix} \mathbf{E}' \\ \mathbf{H}' \\ \mathbf{P}' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \sqrt{2} + 1 & 0 & \sqrt{2} \\ 0 & -(\sqrt{2} + 1) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \\ \mathbf{P} \end{bmatrix}.$$

3.2 First type spacelike curve

Definition 3.6. Let $\gamma : I \rightarrow \mathbb{E}_1^3$ be a spacelike curve with timelike normal depending on the arclength parameter s . This means that the frame $\{\mathbf{E}, \mathbf{H}, \mathbf{P}\}$ is uniquely determined where

$$\mathbf{H} = \gamma'(s), \quad \mathbf{E} = \frac{\gamma''(s)}{\|\gamma''(s)\|} \quad \text{and} \quad \mathbf{P} = -\mathbf{H} \times \mathbf{E} - \mathbf{E}$$

This frame is called the hybrid frame of the spacelike curve with timelike normal. The frame $\{\mathbf{E}, \mathbf{H}, \mathbf{P}\}$ for the spacelike curve is a positively directed basis.

Theorem 3.7. Let $\gamma : I \rightarrow \mathbb{E}_1^3$ be a spacelike curve with timelike normal depending on the arclength parameter s . The hybrid frame formulas are

$$\begin{bmatrix} \mathbf{E}' \\ \mathbf{H}' \\ \mathbf{P}' \end{bmatrix} = \begin{bmatrix} \kappa_2 & \kappa_1 & \kappa_2 \\ \kappa_3 & 0 & \kappa_3 - \kappa_1 \\ 0 & -\kappa_3 & -\kappa_2 \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} -\tau & \kappa & -\tau \\ -\kappa & 0 & -2\kappa \\ 0 & \kappa & \tau \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \\ \mathbf{P} \end{bmatrix}. \quad (2)$$

Proof. The relationship between the hybrid frame vectors $\mathbf{E}, \mathbf{H}, \mathbf{P}$ and the Frenet vectors $\mathbf{T}, \mathbf{N}, \mathbf{B}$ of a spacelike curve with timelike normal can be given as follows:

$$\begin{bmatrix} \mathbf{E} \\ \mathbf{H} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}$$

Therefore, we have

$$\mathbf{E} = \mathbf{N}, \quad \mathbf{H} = \mathbf{T}$$

and

$$\mathbf{B} = \mathbf{H} \times \mathbf{E} = -\mathbf{P} - \mathbf{E} \Rightarrow \mathbf{B} = -\mathbf{P} - \mathbf{N} \Rightarrow \mathbf{P} = -\mathbf{B} - \mathbf{N}.$$

Moreover, we obtain the relationships between the curvatures of the curve as follows:

$$\begin{aligned}\kappa_1 &= \langle \mathbf{E}', \mathbf{H} \rangle = \langle \mathbf{N}', \mathbf{T} \rangle = \langle \kappa \mathbf{T} + \tau \mathbf{B}, \mathbf{T} \rangle = \kappa \\ \kappa_2 &= \langle \mathbf{E}', \mathbf{P} \rangle = \langle \mathbf{N}', -\mathbf{B} - \mathbf{N} \rangle = \langle \kappa \mathbf{T} + \tau \mathbf{B}, -\mathbf{B} - \mathbf{N} \rangle = -\tau \\ \kappa_3 &= \langle \mathbf{H}', \mathbf{P} \rangle = \langle \mathbf{T}', -\mathbf{B} - \mathbf{N} \rangle = \langle \kappa \mathbf{N}, -\mathbf{B} - \mathbf{N} \rangle = -\kappa\end{aligned}$$

Therefore, for a spacelike curve with timelike normal the hybrid frame formulas can be calculated as follows.

If $\mathbf{E}' = a_1 \mathbf{E} + b_1 \mathbf{H} + c_1 \mathbf{P}$, then

$$\begin{aligned}\langle \mathbf{E}', \mathbf{E} \rangle &= \langle a_1 \mathbf{E} + b_1 \mathbf{H} + c_1 \mathbf{P}, \mathbf{E} \rangle = -a_1 + c_1 = 0, \\ \langle \mathbf{E}', \mathbf{P} \rangle &= \langle a_1 \mathbf{E} + b_1 \mathbf{H} + c_1 \mathbf{P}, \mathbf{P} \rangle = a_1 = \kappa_2 = -\tau, \\ \langle \mathbf{E}', \mathbf{H} \rangle &= \langle a_1 \mathbf{E} + b_1 \mathbf{H} + c_1 \mathbf{P}, \mathbf{H} \rangle = b_1 = \kappa_1 = \kappa.\end{aligned}$$

Similar to proof of previous theorem, we can find

$$-a_2 + c_2 = -\kappa_1 = -\kappa, \quad a_2 = \kappa_3 = -\kappa, \quad b_2 = 0,$$

and

$$-a_3 + c_3 = -\kappa_2 = \tau, \quad a_3 = 0, \quad b_3 = -\kappa_3 = \kappa.$$

Thus, we have (2). $\square$

Remark 3.8. Equation (2) shows how the hybrid frame evolves for a spacelike curve with timelike normal in terms of the classical curvature and torsion. The relations $\kappa_1 = \kappa$, $\kappa_2 = -\tau$, and $\kappa_3 = -\kappa$ reflect the change of causal structure within the hybrid formulation.

The geometric configuration of the hybrid frame for a spacelike curve with timelike normal is shown in Figure 2.

Example. Let's construct the frame $\{\mathbf{E}, \mathbf{H}, \mathbf{P}\}$ of the unit speed spacelike curve with timelike normal vector

$$\gamma(s) = \frac{1}{\sqrt{2}}(\cosh s, s, \sinh s)$$

and compute its curvatures and hybrid frame formulas. The vectors of the frame $\{\mathbf{E}, \mathbf{H}, \mathbf{P}\}$ of the curve $\gamma(s)$ are given by

$$\begin{aligned}\mathbf{E} &= (\cosh s, 0, \sinh s), \\ \mathbf{H} &= \frac{\sqrt{2}}{2}(\sinh s, 1, \cosh s),\end{aligned}$$

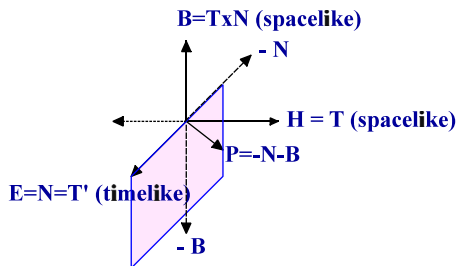


Figure 2: The relationship between the hybrid frame vectors $\mathbf{E}, \mathbf{H}, \mathbf{P}$ and the Frenet vectors $\mathbf{T}, \mathbf{N}, \mathbf{B}$ of a spacelike curve with timelike normal.

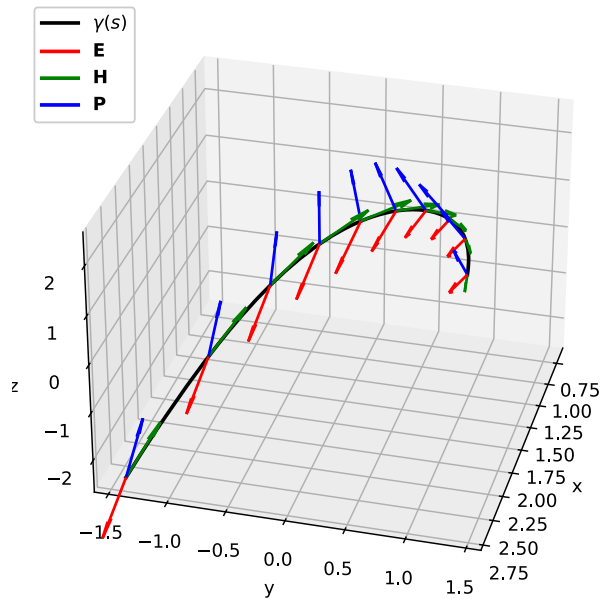


Figure 3: Hybrid frame $\{E, H, P\}$ of the spacelike curve $\gamma(s) = \frac{1}{\sqrt{2}}(\cosh s, s, \sinh s)$ with timelike normal vector. The frame vectors are illustrated at five selected parameter values along the curve.

$$\begin{aligned} P &= -H \times E - E = -\frac{\sqrt{2}}{2}(-\sinh s, 1, -\cosh s) - (\cosh s, 0, \sinh s) \\ &= \left(\frac{\sqrt{2}}{2} \sinh s - \cosh s, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \cosh s - \sinh s \right) \end{aligned}$$

The curvatures of the curve $\gamma(s)$ are found as follows:

$$\kappa_1 = \langle E', H \rangle = \frac{1}{\sqrt{2}}$$

$$\kappa_2 = \langle E', P \rangle = \frac{1}{\sqrt{2}}$$

$$\kappa_3 = \langle H', P \rangle = \frac{1}{\sqrt{2}}$$

The hybrid frame formulas are found as follows:

$$\begin{bmatrix} E' \\ H' \\ P' \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} E \\ H \\ P \end{bmatrix}.$$

The geometric behavior of the hybrid frame along the curve is illustrated in Figure 3.

3.3 Second type spacelike curve

Definition 3.9. Let $\gamma : I \rightarrow \mathbb{E}_1^3$ be a spacelike curve with timelike binormal depending on the arclength parameter s . This means that the frame $\{E, H, P\}$ is uniquely determined where

$$H = T = \gamma', \quad E = B = \frac{\gamma' \times \gamma''}{\|\gamma' \times \gamma''\|} \quad \text{and} \quad P = N - B.$$

This frame is called the hybrid frame of the spacelike curve with timelike binormal. The frame $\{\mathbf{E}, \mathbf{H}, \mathbf{P}\}$ for the spacelike curve is a positively directed basis.

Theorem 3.10. *Let $\gamma : I \rightarrow \mathbb{E}_1^3$ be a spacelike curve with timelike binormal depending on the arclength parameter s . The hybrid frame formulas are*

$$\begin{bmatrix} \mathbf{E}' \\ \mathbf{H}' \\ \mathbf{P}' \end{bmatrix} = \begin{bmatrix} \kappa_2 & 0 & \kappa_2 \\ \kappa_3 & 0 & \kappa_3 \\ 0 & -\kappa_3 & -\kappa_2 \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \tau & 0 & \tau \\ \kappa & 0 & \kappa \\ 0 & -\kappa & -\tau \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \\ \mathbf{P} \end{bmatrix} \quad (3)$$

Proof. The relationship between the hybrid frame vectors $\mathbf{E}, \mathbf{H}, \mathbf{P}$ and the Frenet vectors $\mathbf{T}, \mathbf{N}, \mathbf{B}$ of a spacelike curve with timelike binormal can be given as follows:

$$\begin{bmatrix} \mathbf{E} \\ \mathbf{H} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}$$

Therefore, we obtain

$$\mathbf{E} = \mathbf{B}, \quad \mathbf{H} = \mathbf{T}$$

and

$$\begin{aligned} \mathbf{B} \times \mathbf{T} = \mathbf{N} &\Rightarrow \mathbf{E} \times \mathbf{H} = \mathbf{N} \Rightarrow \mathbf{P} + \mathbf{E} = \mathbf{N} \\ &\Rightarrow \mathbf{P} + \mathbf{B} = \mathbf{N} \Rightarrow \mathbf{P} = \mathbf{N} - \mathbf{B} = (\mathbf{E} \times \mathbf{H}) - \mathbf{E}. \end{aligned}$$

Also, the relationship between the curvatures of the curve γ is found as

$$\begin{aligned} \kappa_1 &= \langle \mathbf{E}', \mathbf{H} \rangle = \langle \mathbf{B}', \mathbf{T} \rangle = \langle \tau \mathbf{N}, \mathbf{T} \rangle = 0, \\ \kappa_2 &= \langle \mathbf{E}', \mathbf{P} \rangle = \langle \mathbf{B}', \mathbf{N} - \mathbf{B} \rangle = \langle \tau \mathbf{N}, \mathbf{N} - \mathbf{B} \rangle = \tau, \\ \kappa_3 &= \langle \mathbf{H}', \mathbf{P} \rangle = \langle \mathbf{T}', \mathbf{N} - \mathbf{B} \rangle = \langle \kappa \mathbf{N}, \mathbf{N} - \mathbf{B} \rangle = \kappa. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} -a_1 + c_1 &= 0, \quad a_1 = \kappa_2 = \tau, \quad b_1 = \kappa_1 = 0, \\ -a_2 + c_2 &= -\kappa_1 = 0, \quad a_2 = \kappa_3 = \kappa, \quad b_2 = 0, \\ -a_3 + c_3 &= -\kappa_2 = -\tau, \quad a_3 = 0, \quad b_3 = -\kappa_3 = -\kappa. \end{aligned}$$

Thus we have (3). □

Remark 3.11. Equation (3) shows that for a spacelike curve with timelike binormal, the hybrid frame evolution is determined by $\kappa_2 = \tau$ and $\kappa_3 = \kappa$, while $\kappa_1 = 0$. This reflects the change in the causal configuration of the curve within the hybrid formulation.

Figure 4 illustrates the hybrid frame vectors for spacelike curves with timelike binormal.

Example. Let's construct the frame $\{\mathbf{E}, \mathbf{H}, \mathbf{P}\}$ of the unit speed spacelike curve with spacelike normal vector

$$\gamma(s) = (s, \cos \sqrt{2}s, \sin \sqrt{2}s)$$

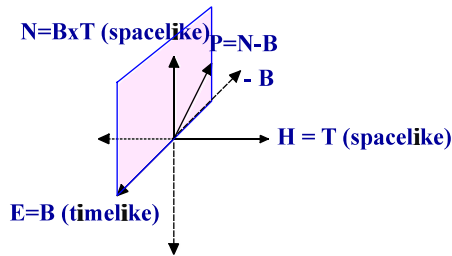


Figure 4: The relationship between the hybrid frame vectors $\mathbf{E}$, $\mathbf{H}$, $\mathbf{P}$ and the Frenet vectors $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ of a spacelike curve with timelike binormal.

and compute its curvatures and hybrid frame formulas. The vectors of the frame $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{P}$ of the curve $\gamma(s)$ are

$$\begin{aligned}\mathbf{E} &= (-\sqrt{2}, \sin \sqrt{2}s, -\cos \sqrt{2}s), \\ \mathbf{H} &= (1, -\sqrt{2} \sin \sqrt{2}s, \sqrt{2} \cos \sqrt{2}s), \\ \mathbf{P} &= \mathbf{E} \times \mathbf{H} - \mathbf{E} = (0, \cos \sqrt{2}s, \sin \sqrt{2}s) - (-\sqrt{2}, \sin \sqrt{2}s, -\cos \sqrt{2}s) \\ &= (\sqrt{2}, \cos \sqrt{2}s - \sin \sqrt{2}s, \cos \sqrt{2}s + \sin \sqrt{2}s)\end{aligned}$$

The curvatures of the curve $\gamma(s)$ are found as follows:

$$\kappa_1 = \langle \mathbf{E}', \mathbf{H} \rangle = 0, \quad \kappa_2 = \langle \mathbf{E}', \mathbf{P} \rangle = \sqrt{2} \text{ and } \kappa_3 = \langle \mathbf{H}', \mathbf{P} \rangle = -2.$$

The hybrid frame formulas are found as follows:

$$\begin{bmatrix} \mathbf{E}' \\ \mathbf{H}' \\ \mathbf{P}' \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 & \sqrt{2} \\ -2 & 0 & -2 \\ 0 & 2 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \\ \mathbf{P} \end{bmatrix}.$$

The corresponding hybrid frame representation is shown in Figure 5.

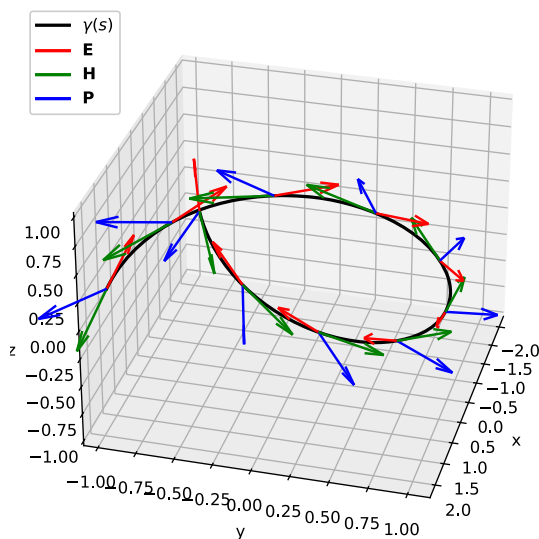


Figure 5: Hybrid frame $\{\mathbf{E}, \mathbf{H}, \mathbf{P}\}$ of the unit speed spacelike curve $\gamma(s) = (s, \cos(\sqrt{2}s), \sin(\sqrt{2}s))$ with spacelike normal vector. The frame vectors are illustrated at ten selected parameter values along the curve.

4 Osculating sphere of a curve with the frame $\{\mathbf{E}, \mathbf{H}, \mathbf{P}\}$

Geometric constructions associated with non-lightlike curves, including tubular and directional surfaces, have also been investigated in Lorentzian settings; see for instance Dede et al. [19].

Definition 4.1. The osculating sphere of γ at $\gamma(t)$ is the sphere having a fourth-order contact with the curve at that point.

This definition applies when the curvature and torsion of the curve are nonzero [20].

Theorem 4.2. Let $\gamma : I \rightarrow \mathbb{E}_1^3$ be a unit speed timelike curve with the frame $\{\mathbf{E}, \mathbf{H}, \mathbf{P}\}$. Take the function $G(t)$ as

$$G(t) = \langle \gamma(t) - \mathbf{M}, \gamma(t) - \mathbf{M} \rangle \pm \rho^2$$

where $\mathbf{M}$ is the center and ρ is the radius of the osculating sphere of γ at $\gamma(t)$. Then the center of the osculating sphere is

$$\mathbf{M}(t) = \gamma(t) + \frac{\kappa_1'}{(\kappa_3 - \kappa_1)\kappa_1^2} \mathbf{E} - \frac{1}{\kappa_1} \mathbf{H} + \frac{\kappa_1'}{(\kappa_3 - \kappa_1)\kappa_1^2} \mathbf{P}$$

and the radius of the osculating sphere is

$$\rho(t) = \frac{1}{\kappa_1} \sqrt{\left(1 + \frac{(\kappa_1')^2}{(\kappa_3 - \kappa_1)^2 \kappa_1^2}\right)}.$$

Proof. We know that the curve γ and its osculating sphere have at least third-order contact. So we write

$$G(t) = G'(t) = G''(t) = G'''(t) = 0.$$

Thus, we have $\gamma(t) - \mathbf{M} = x\mathbf{E} + y\mathbf{H} + z\mathbf{P}$. If $\mathbf{E}' = \kappa_1\mathbf{H}$ and $\mathbf{H}' = (\kappa_1 + (\kappa_3 - \kappa_1))\mathbf{E} + (\kappa_3 - \kappa_1)\mathbf{P}$ are used for a timelike curve, we obtain

$$\langle \mathbf{E}, \gamma(t) - \mathbf{M} \rangle = z - x$$

$$\langle \mathbf{H}, \gamma(t) - \mathbf{M} \rangle = y$$

$$\langle \mathbf{P}, \gamma(t) - \mathbf{M} \rangle = x.$$

From these equalities, we obtain

$$G'(t) = 2(\langle \mathbf{E}, \gamma(t) - \mathbf{M} \rangle) \Rightarrow 0 = z - x \Rightarrow x = z$$

$$G''(t) = 2(\langle \kappa_1\mathbf{H}, \gamma(t) - \mathbf{M} \rangle - 1) \Rightarrow 0 = \kappa_1 y - 1 \Rightarrow y = \frac{1}{\kappa_1}$$

$$G'''(t) = 2(\kappa_1' \langle \mathbf{H}, \gamma(t) - \mathbf{M} \rangle + \kappa_1 \langle (\kappa_1 + (\kappa_3 - \kappa_1))\mathbf{E}, \gamma(t) - \mathbf{M} \rangle + \kappa_1 \langle (\kappa_3 - \kappa_1)\mathbf{P}, \gamma(t) - \mathbf{M} \rangle + \kappa_1 \langle \mathbf{H}, \mathbf{E} \rangle)$$

$$G'''(t) = (\kappa_1' y + \kappa_1 (\kappa_1 + (\kappa_3 - \kappa_1))(z - x) + \kappa_1 (\kappa_3 - \kappa_1)x)$$

$$G'''(t) = \left(\frac{\kappa_1'}{\kappa_1} + \kappa_1^2 z - \kappa_1^2 x + \kappa_1 (\kappa_3 - \kappa_1)z - \kappa_1 (\kappa_3 - \kappa_1)x + \kappa_1 (\kappa_3 - \kappa_1)x \right)$$

$$0 = \kappa_1' \frac{1}{\kappa_1} + \kappa_1 (\kappa_3 - \kappa_1)z$$

$$z = -\frac{\kappa_1'}{(\kappa_3 - \kappa_1)\kappa_1^2}$$

Then, the radius ρ of the osculating sphere of the curve γ at the point $\gamma(t)$ and the coordinates of its center $\mathbf{M}$ are found as follows:

$$\begin{aligned}\mathbf{M} &= \gamma(t) - x\mathbf{E} - y\mathbf{H} - z\mathbf{P} \\ &= \gamma(t) + \frac{\kappa'_1}{(\kappa_3 - \kappa_1)\kappa_1^2}\mathbf{E} - \frac{1}{\kappa_1}\mathbf{H} + \frac{\kappa'_1}{(\kappa_3 - \kappa_1)\kappa_1^2}\mathbf{P}\end{aligned}$$

and

$$\begin{aligned}\rho^2 &= \langle \gamma(t) - \mathbf{M}, \gamma(t) - \mathbf{M} \rangle \\ \rho^2 &= \left\langle -\frac{\kappa'_1\mathbf{E}}{(\kappa_3 - \kappa_1)\kappa_1^2} + \frac{\mathbf{H}}{\kappa_1} - \frac{\kappa'_1\mathbf{P}}{(\kappa_3 - \kappa_1)\kappa_1^2}, -\frac{\kappa'_1\mathbf{E}}{(\kappa_3 - \kappa_1)\kappa_1^2} + \frac{\mathbf{H}}{\kappa_1} - \frac{\kappa'_1\mathbf{P}}{(\kappa_3 - \kappa_1)\kappa_1^2} \right\rangle \\ \rho^2 &= -\left(\frac{\kappa'_1}{(\kappa_3 - \kappa_1)\kappa_1^2}\right)^2 + \left(\frac{\kappa'_1}{(\kappa_3 - \kappa_1)\kappa_1^2}\right)^2 + \frac{1}{\kappa_1^2} + \left(\frac{\kappa'_1}{(\kappa_3 - \kappa_1)\kappa_1^2}\right)^2 \\ \rho^2 &= \frac{1}{\kappa_1^2} \left(1 + \frac{(\kappa'_1)^2}{(\kappa_3 - \kappa_1)^2 \kappa_1^2}\right) \\ \rho &= \frac{1}{\kappa_1} \sqrt{\left(1 + \frac{(\kappa'_1)^2}{(\kappa_3 - \kappa_1)^2 \kappa_1^2}\right)},\end{aligned}$$

respectively. □

Remark 4.3. The above result expresses the center of the osculating sphere in terms of the hybrid frame vectors and the corresponding curvature functions. This formulation describes the local behavior of the curve by incorporating elliptic, hyperbolic, and parabolic directions within the hybrid setting. In particular, the osculating sphere is characterized directly through the hybrid invariants of the curve.

Similarly, one can obtain analogous results for spacelike curves with timelike or spacelike normals, which yield the following theorems.

Theorem 4.4. Let $\gamma : I \rightarrow \mathbb{E}_1^3$ be a unit speed spacelike curve having timelike normals with the frame $\{\mathbf{E}, \mathbf{H}, \mathbf{P}\}$. Take the function $G(t)$ as

$$G(t) = \langle \gamma(t) - \mathbf{M}, \gamma(t) - \mathbf{M} \rangle \pm \rho^2$$

where $\mathbf{M}$ is the center and ρ is the radius of the osculating sphere of γ at $\gamma(t)$. Then we obtain

$$\begin{aligned}\mathbf{M} &= \gamma(t) - \frac{\kappa_1(-\kappa_2) - \kappa'_1}{3\kappa_1^2(-\kappa_2)}\mathbf{E} - \frac{2\kappa_1(-\kappa_2) + \kappa'_1}{3\kappa_1^2(-\kappa_2)}\mathbf{P}, \\ \rho(t) &= \sqrt{-\left(\frac{\kappa_1(-\kappa_2) - \kappa'_1}{3\kappa_1^2(-\kappa_2)}\right)^2 - 2\frac{2\kappa_1(-\kappa_2) + \kappa'_1}{3\kappa_1^2(-\kappa_2)}\frac{\kappa_1(-\kappa_2) - \kappa'_1}{3\kappa_1^2(-\kappa_2)}}\end{aligned}$$

Theorem 4.5. Let $\gamma : I \rightarrow \mathbb{E}_1^3$ be a unit speed spacelike curve having spacelike normals with the frame $\{\mathbf{E}, \mathbf{H}, \mathbf{P}\}$. Take the function $G(t)$ as

$$G(t) = \langle \gamma(t) - \mathbf{M}, \gamma(t) - \mathbf{M} \rangle \pm \rho^2,$$

where $\mathbf{M}$ is the center and ρ is the radius of the osculating sphere of γ at $\gamma(t)$. Then we obtain

$$\mathbf{M}(t) = \gamma(t) + \frac{\kappa'_3 + \kappa_2 \kappa_3}{\kappa_3^2 \kappa_2} \mathbf{E} + \frac{1}{\kappa_3} \mathbf{P},$$

$$\rho(t) = \sqrt{\left(\frac{\kappa'_3 + \kappa_2 \kappa_3}{\kappa_3^2 \kappa_2}\right)^2 + 2 \frac{\kappa'_3 + \kappa_2 \kappa_3}{\kappa_3^3 \kappa_2}}.$$

To illustrate Theorem 4.4, we compute the center and the radius of the osculating sphere for a specific curve.

Example. Consider the unit speed spacelike curve

$$\gamma(s) = \frac{1}{\sqrt{2}}(\cosh s, s, \sinh s),$$

for which

$$\kappa_1 = \kappa_2 = \kappa_3 = \frac{1}{\sqrt{2}}$$

and $\kappa'_1 = 0$. Substituting these values into the expressions given in Theorem 4.4, we first compute

$$\kappa_1(-\kappa_2) - \kappa'_1 = -\frac{1}{2}, \quad 3\kappa_1^2(-\kappa_2) = -\frac{3}{2\sqrt{2}},$$

which gives

$$\frac{\kappa_1(-\kappa_2) - \kappa'_1}{3\kappa_1^2(-\kappa_2)} = \frac{\sqrt{2}}{3}.$$

Similarly,

$$\frac{2\kappa_1(-\kappa_2) + \kappa'_1}{3\kappa_1^2(-\kappa_2)} = \frac{2\sqrt{2}}{3}.$$

Hence, the center of the osculating sphere is

$$\mathbf{M}(s) = \gamma(s) - \frac{\sqrt{2}}{3} \mathbf{E}(s) - \frac{2\sqrt{2}}{3} \mathbf{P}(s).$$

For the radius, we substitute the same values into the formula of Theorem 4.4. Since $\kappa'_1 = 0$, we obtain

$$\rho = \sqrt{-\left(\frac{\sqrt{2}}{3}\right)^2 - 2\left(\frac{2\sqrt{2}}{3}\right)\left(\frac{\sqrt{2}}{3}\right)} = \sqrt{\frac{10}{9}} = \frac{\sqrt{10}}{3}.$$

5 Conclusions

In this paper, we introduced a hybrid frame $\{\mathbf{E}, \mathbf{H}, \mathbf{P}\}$ constructed from elliptic, hyperbolic, and parabolic vectors for studying non-lightlike curves in Minkowski 3-space. We defined the curvature and torsion of these curves within the hybrid framework and compared the obtained results with the classical Frenet formulation. In addition, we examined the osculating spheres of non-lightlike curves and expressed their centers in terms of the hybrid frame and its curvature functions. The results show that the hybrid frame provides an alternative geometric description that incorporates different causal vector types in a unified manner.

The present study may be extended in several natural directions. In particular, the hybrid frame formulation can be adapted to the investigation of lightlike curves in Minkowski 3-space. Another possible direction is a deeper comparison between the hybrid frame and other alternative frame constructions in Lorentzian

geometry. Furthermore, a more detailed analysis of the global geometric properties of curves within the hybrid framework may contribute to a better understanding of its structural features.

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